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An abstract transmission problem in a thin layer, I: Sharp estimates

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Abstract

We study an elliptic transmission problem in Banach spaces. The problem is considered on the juxtaposition of two intervals, one of which of small length δ, and models physical phenomena in media constituted by two parts with different physical characteristics. We obtain results of existence, uniqueness, maximal regularity and optimal dependence on the parameter δ for L^p solutions of the problem. The main tools of our approach are impedance and admittance operators (i.e. Dirichlet-to-Neumann and Neumann-to-Dirichlet operators) and H^∞ functional calculus for sectorial operators in Banach spaces.

Keywords: Abstract differential equations; Transmission problem; H^∞ functional calculus; Impedance operator

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1. Introduction

In this paper we study the abstract transmission problem

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[, \\ u_+''(x) + Au_+(x) = -g_+(x), & x \in]0,\delta[, \\ u_-(-1) = f_-, \\ u_+'(\delta) = f_+, \\ u_-(0) = u_+(0), \\ p_-u_-'(0) = p_+u_+'(0) + \chi, \end{cases} \tag{ATP}$$

where A is a linear closed operator in a Banach space X, $p_-, p_+ \in \mathbb{R}^+$, δ is a small parameter, $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_+, f_-, \chi \in X$.

We are interested in the existence and uniqueness of the solution (u_-, u_+) of this problem in the sense of L^p spaces and in its behaviour for small values of the parameter δ.

We call L^p solution of problem (ATP) a pair of functions (u_-, u_+) with $u_- \in W^{2,p}(]-1,0[;X) \cap L^p(]-1,0[;\mathcal{D}(A))$, $u_+ \in W^{2,p}(]0,\delta[;X) \cap L^p(]0,\delta[;\mathcal{D}(A))$, such that $u_-''(x) + Au_-(x) = -g_-(x)$ for a.e. $x \in]-1,0[$, $u_+''(x) + Au_+(x) = -g_+(x)$ for a.e. $x \in]0,\delta[$ and the equalities $u_-(-1) = f_-$, $u_+'(\delta) = f_+$, $u_-(0) = u_+(0)$, $p_-u_-'(0) = p_+u_+'(0) + \chi$ are satisfied.

We study this problem separately on the two intervals $]-1,0[$ and $]0,\delta[$. To this end we search for a limit condition at 0 for u_- that replace the equation

$$u_+''(x) + Au_+(x) = -g_+(x), \quad x \in]0,\delta[,$$

and the limit and transmission conditions

$$u_+'(\delta) = f_+, \qquad u_-(0) = u_+(0), \qquad p_-u_-'(0) = p_+u_+'(0) + \chi.$$

This is possible by means of the notion of impedance.

We call impedance (or Dirichlet-to-Neumann) operator on the thin layer $]0,\delta[$ the operator

$$T_{0,\delta} : (g_+, f_+, \psi) \mapsto T_{0,\delta}(g_+, f_+, \psi) = u_+'(0)$$

where u_+ is the solution of the limit problem

$$\begin{cases} u_+''(x) + Au_+(x) = -g_+(x), & x \in]0,\delta[, \\ u_+'(\delta) = f_+, \\ u_+(0) = \psi. \end{cases}$$

From the transmission conditions $u_+(0) = u_-(0)$ and $p_-u_-'(0) = p_+u_+'(0) + \chi$ it follows

$$p_-u_-'(0) = p_+T_{0,\delta}\big(g_+, f_+, u_-(0)\big) + \chi$$

hence we can bring back the transmission problem to a limit problem on the interval $]-1,0[$ (limit problem with impedance condition):

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[, \\ u_-(-1) = f_-, \\ p_-u_-'(0) = p_+T_{0,\delta}(g_+, f_+, u_-(0)) + \chi. \end{cases} \tag{LPIC}$$

The limit condition at 0: $p_-u_-'(0) = p_+T_{0,\delta}(g_+, f_+, u_-(0)) + \chi$ (called impedance condition) takes into account the effect of the thin layer $]0,\delta[$.

Once problem (LPIC) is solved, we have the following problem on $]0,\delta[$ for u_+ (thin layer problem):

$$\begin{cases} u_+''(x) + Au_+(x) = -g_+(x), & x \in]0,\delta[, \\ u_+(0) = u_-(0), \\ u_+'(\delta) = f_+. \end{cases} \tag{TLP}$$

Therefore we prove existence, uniqueness and estimates for the L^p solution of problem (ATP) through the study of problems (LPIC) and (TLP), whose properties are stated in Theorems 6.1 and 7.1.

Problem (ATP) is studied in [17] in the L^p setting and in [3] and [4] in Hölder spaces. In [17] the authors use a change of scale and the method of the sum of operators of Da Prato and Grisvard; the non-homogeneous term belongs to an interpolation space between L^p and a Sobolev space. In [3] and [4] the authors use an explicit representation of the solution through a Dunford integral containing the Green kernel; the non-homogeneous term belongs to a Hölder space. The papers [1,2,5,24,25] study thin layer problems in an Hilbertian framework; one of the key tools is an asymptotic expansion with respect to the parameter δ.

In this paper we work in an L^p framework ($1 < p < \infty$), hence with L^p non-homogeneous term. We represent the solution of problem (ATP) by means of the semigroup generated by the operator $-(-A)^{1/2}$. In order to estimate this solution we exploit the H^∞ functional calculus for the sectorial operator $(-A)^{1/2}$ and the fact that L^p is a UMD space (or, equivalently, the fact that the Hilbert transform behaves well in such a space). We prove maximal regularity of the solution in Sobolev spaces with estimates that are sharp with regard to the dependence on the parameters δ, p_- and p_+.

In a forthcoming paper we shall study the limit problem for $\delta \to 0$.

The organisation of the paper is the following. In Section 2 we describe three physical models of the problem studied in the paper. In Section 3 we list the results concerning H^∞ functional calculus on which the paper is based. In Section 4 we list some preliminary results. In Section 5 we study the impedance and admittance operators. In Sections 6 and 7 we state and prove the main results of the paper, concerning existence, uniqueness and estimates of L^p solutions for the transmission problem. In Section 8 we give a series of examples showing that the estimates obtained in Sections 6 and 7 are sharp. In Section 9 we give two examples of application of the results of the paper.

2. Physical models

In this section we introduce some physical examples, motivating the interest of problem ATP.

Example 2.1 *(Steady state heat flux in a heterogeneous stem).* We consider the problem of steady state heat flux in a stem constituted by the junction of two stems $]-1,0[$ and $]0,\delta[$ of different physical characteristics, with the condition that the temperature and the heat flux are continuous at the joining of the stems.

We consider the functions u_-, defined on $]-1,0[$, and u_+, defined on $]0,\delta[$, such that $u_\pm(x)$ is the temperature at the point x; if p_- and p_+ are the coefficients of conductivity of the stems $]-1,0[$ and $]0,\delta[$ respectively, $a>0$ is a coefficient of exchange with the exterior, $p_\pm g_\pm(x)$ is the heat source at the point x, the problem can be modelled by the equation $u''_\pm(x)-au_\pm(x)=-g_\pm(x)$ and the conditions at the joining are $u_-(0)=u_+(0)$ and $p_-u'_-(0)=p_+u'_+(0)$. By adding the condition that the temperature at -1 is f_- and the heat flux at δ is p_+f_+, we obtain the following transmission problem:

$$\begin{cases} u''_-(x)-au_-(x)=-g_-(x), & x\in\,]-1,0[,\\ u''_+(x)-au_+(x)=-g_+(x), & x\in\,]0,\delta[,\\ u_-(-1)=f_-,\\ u'_+(\delta)=f_+,\\ u_-(0)=u_+(0),\\ p_-u'_-(0)=p_+u'_+(0). \end{cases}$$

Example 2.2 *(Steady state heat flux in a heterogeneous shell).* We consider the problem of steady state heat flux in a cylindrical shell constituted by the junction of two homogeneous cylindrical shells; the points on the shells are described by the coordinates (x,y) with $x\in\,]-1,0[$ for one shell and $x\in[0,\delta]$ for the other one and $y\in[-\pi,\pi]$ (obviously $y=-\pi$ and $y=\pi$ identify the same point).

If $u_\pm(x,y)$ is the temperature at the point (x,y), p_- and p_+ are the coefficients of conductivity of the shells $]-1,0[\times[-\pi,\pi]$ and $]0,\delta[\times[-\pi,\pi]$ respectively, $p_\pm g_\pm(x,y)$ is the heat source at the point (x,y), $f_-(y)$ is the temperature at the point $(-1,y)$ and $p_+f_+(y)$ is the heat flux at the point (δ,y), the heat flux is described by the following system:

$$\begin{cases} \Delta u_-(x,y)=-g_-(x,y), & (x,y)\in\,]-1,0[\times]-\pi,\pi[,\\ \Delta u_+(x,y)=-g_+(x,y), & (x,y)\in\,]0,\delta[\times]-\pi,\pi[,\\ u_-(-1,y)=f_-(y), & y\in\,]-\pi,\pi[,\\ \dfrac{\partial u_+}{\partial x}(\delta,y)=f_+(y), & y\in\,]-\pi,\pi[,\\ u_-(x,\pi)=u_-(x,-\pi), & x\in\,]-1,0[,\\ u_+(x,\pi)=u_+(x,-\pi), & x\in\,]0,\delta[,\\ \dfrac{\partial u_-}{\partial x}(x,\pi)=\dfrac{\partial u_-}{\partial x}(x,-\pi), & x\in\,]-1,0[,\\ \dfrac{\partial u_+}{\partial x}(x,\pi)=\dfrac{\partial u_+}{\partial x}(x,-\pi), & x\in\,]0,\delta[,\\ u_-(0,y)=u_+(0,y), & y\in\,]-\pi,\pi[,\\ p_-\dfrac{\partial u_-}{\partial x}(0,y)=p_+\dfrac{\partial u_+}{\partial x}(0,y), & y\in\,]-\pi,\pi[. \end{cases}$$

Let X be the space of 2π-periodic functions, with locally summable p-th power; if we consider the X-valued functions $v_\pm$ such that $v_\pm(x)(y) = u_\pm(x,y)$ the transmission problem can be written in the following abstract form:

$$\begin{cases} v_-''(x) + Av_-(x) = -g_-(x), & x \in]-1,0[, \\ v_+''(x) + Av_+(x) = -g_+(x), & x \in]0,\delta[, \\ v_-(-1) = f_-, \\ v_+'(\delta) = f_+, \\ v_-(0) = v_+(0), \\ p_- v_-'(0) = p_+ v_+'(0), \end{cases}$$

where A is the operator with domain $\mathcal{D}(A) = \{v \in X\colon\ v'' \in X\}$ and values in X such that $Av = v''$.

Example 2.3 *(Steady state heat flux in a heterogeneous cylinder).* We consider the problem of steady state heat flux in a cylinder $]-1,\delta[\times \Omega$, constituted by the junction of two homogeneous cylinders $]-1,0[\times \Omega$ and $]0,\delta[\times \Omega$ (Ω open bounded set of $\mathbb{R}^2$ with regular boundary Γ) of different physical characteristics.

If we suppose that the temperature vanishes on the lateral surface of the cylinder, the heat flux can be modelled by the following transmission problem:

$$\begin{cases} \Delta u_-(x,y) = -g_-(x,y), & (x,y) \in]-1,0[\times \Omega, \\ \Delta u_+(x,y) = -g_+(x,y), & (x,y) \in]0,\delta[\times \Omega, \\ u_-(-1,y) = f_-(y), & y \in \Omega, \\ \dfrac{\partial u_+}{\partial x}(\delta,y) = f_+(y), & y \in \Omega, \\ u_-(x,y) = 0, & x \in]-1,0[,\ y \in \Gamma, \\ u_+(x,y) = 0, & x \in]0,\delta[,\ y \in \Gamma, \\ u_-(0,y) = u_+(0,y), & y \in \Omega, \\ p_- \dfrac{\partial u_-}{\partial x}(0,y) = p_+ \dfrac{\partial u_+}{\partial x}(0,y), & y \in \Omega, \end{cases}$$

where $u_\pm(x,y)$ is the temperature at the point (x,y), p_- and p_+ are the coefficients of conductivity of the cylinders $]-1,0[\times \Omega$ and $]0,\delta[\times \Omega$ respectively, $p_\pm g_\pm(x,y)$ is the source of heat at the point (x,y), $f_-(y)$ is the temperature at the point $(-1,y)$ and $p_+ f_+(y)$ is the heat flux at the point (δ,y).

Let $X = L^p(\Omega)$ and let us denote by $v_\pm$ the X-valued functions such that $v_\pm(x)(y) = u_\pm(x,y)$. If A is the unbounded operator in $L^p(\Omega)$ such that

$$\begin{cases} \mathcal{D}(A) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \\ A\varphi = \Delta\varphi. \end{cases}$$

The transmission problem can be written in the abstract form

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[, \\ u_+''(x) + Au_+(x) = -g_+(x), & x \in]0,\delta[, \\ u_-(-1) = f_-, \\ u_+'(\delta) = f_+, \\ u_-(0) = u_+(0), \\ p_-u_-'(0) = p_+u_+'(0). \end{cases}$$

Example 2.4 *(Electrostatic potential in a heterogeneous cylinder).* Let $G^\delta =]-1,\delta[\times \Omega$ be the cylinder (of dielectric material) constituted by the junction of two homogeneous cylinders $G_- =]-1,0[\times \Omega$ and $G_+^\delta =]0,\delta[\times \Omega$ (the thin layer) with different electric characteristics; here Ω is a bounded domain of $\mathbb{R}^2$ with regular boundary Γ; we denote by (x,y) the generic variable in G^δ.

The transmission problem:

$$\begin{cases} \dfrac{1}{p}\nabla\cdot(p\nabla u) = -g & \text{in } G^\delta, \\ u = 0 & \text{in }]-1,\delta[\times \Gamma, \\ u = f_- & \text{in } \{-1\} \times \Omega, \\ p\dfrac{\partial u}{\partial x} = f_+ & \text{in } \{\delta\} \times \Omega, \end{cases}$$

is a model for an electrostatic problem in G. The function u is the electrostatic potential, p is the conductivity coefficient, hence $-\nabla u$ is the electric field and $-p\nabla u$ is the electric induction field. The heterogeneity of the material involve the discontinuity of the conductivity coefficient; we have:

$$p = \begin{cases} p_- & \text{in }]-1,0[\times \Omega, \\ p_+ & \text{in }]0,\delta[\times \Omega. \end{cases}$$

The function g is a given electric density, f_- is a fixed surface potential and f_+ a fixed surface induction. If we set

$$u_- = u_{|]-1,0[\times\Omega}, \qquad g_- = g_{|]-1,0[\times\Omega}, \qquad u_+ = u_{|]0,\delta[\times\Omega}, \qquad g_+ = g_{|]0,\delta[\times\Omega},$$

then the equation

$$\frac{1}{p}\nabla\cdot(p\nabla u) = -g, \quad \text{in } G^\delta,$$

is equivalent to the equations

$$\begin{cases} \Delta u_- = -g_- & \text{in }]-1,0[\times \Omega, \\ \Delta u_+ = -g_+ & \text{in }]0,\delta[\times \Omega, \end{cases}$$

with the transmission conditions

$$u_- = u_+, \qquad p_- \frac{\partial u_-}{\partial x} = p_+ \frac{\partial u_+}{\partial x},$$

on $\{0\} \times \Omega$; these conditions represent respectively the continuity across the interface between the two parts of the cylinder of the electrostatic potential and the continuity of the normal component of the electric induction.

This problem can be written in an abstract form analogous to that of Example 2.3.

The presence of the parameter δ raises numerical instability and numerical locking when computing the solution. In the following we make a detailed analysis of the impedance problem (that takes into account the effect of the thin layer), this analysis brings to replace the initial problem with new problems that are well posed. These problems can be treated numerically more efficiently.

3. Functional calculus

The results of this paper are based on the H^∞ functional calculus for sectorial operators that we are going to define. This concept has been introduced in [26] and [7] (see also [23] for a different definition). A complete treatment of the argument can be found in the book [20]. Complete proofs of some of the results quoted here (but in a more general setting) can be found in [12].

We put

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\}\colon\ |\arg z| < \alpha\} \quad \text{for } \alpha \in]0, \pi[,$$
$$\overline{S}_\alpha = \{z \in \mathbb{C} \setminus \{0\}\colon\ |\arg z| \leqslant \alpha\} \cup \{0\} \quad \text{for } \alpha \in [0, \pi[.$$

In the following we shall make use of the fact that, chosen $\alpha \in]0, \pi/2[$,

$$z \in S_\alpha \quad \Longleftrightarrow \quad 0 \leqslant \frac{|\operatorname{Im} z|}{\operatorname{Re} z} < \tan \alpha.$$

Let X be a complex Banach space and A a closed linear densely defined operator in X. We denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\rho(A)$ and $\sigma(A)$ the domain, the range, the resolvent set and the spectrum of A respectively. On the space $\mathcal{D}(A)$ we define the usual norm $\|y\|_{\mathcal{D}(A)} = \|Ay\|_X + \|y\|_X$, where the term $\|y\|_X$ can be omitted if A has bounded inverse.

We say that A is a sectorial operator with spectral angle $\beta \in [0, \pi[$ if $\sigma(A) \subseteq \overline{S}_\beta$ and for each $\alpha \in]\beta, \pi[$,

$$\sup_{w \in \mathbb{C} \setminus S_\alpha} \left\| w(wI - A)^{-1} \right\| < \infty.$$

We note that if $\mathbb{R}^- \subseteq \rho(A)$ and $\sup_{\lambda \in \mathbb{R}^+} \lambda \|(\lambda I + A)^{-1}\| < \infty$ then A is sectorial (see [16], Lemma 6.4.1).

For $\varphi \in]0, \pi[$ we denote by $H(S_\varphi)$ the space of holomorphic functions on S_φ with values in $\mathbb{C}$. Moreover we consider the following subspaces of $H(S_\varphi)$:

$$H^\infty(S_\varphi) = \Big\{ f \in H(S_\varphi)\colon \sup_{z\in S_\varphi} \big|f(z)\big| < \infty \Big\},$$

$$H_0^\infty(S_\varphi) = \Big\{ f \in H(S_\varphi)\colon \exists s \in \mathbb{R}^+ \sup_{z\in S_\varphi} \max\big\{|z|^s, |z|^{-s}\big\}\big|f(z)\big| < \infty \Big\},$$

$$H_P(S_\varphi) = \Big\{ f \in H(S_\varphi)\colon \exists s \in \mathbb{R}^+ \sup_{z\in S_\varphi} \min\big\{|z|^s, |z|^{-s}\big\}\big|f(z)\big| < \infty \Big\},$$

that is $H^\infty(S_\varphi)$ is the space of bounded functions, $H_0^\infty(S_\varphi)$ is the space of the functions decreasing like a power at 0 and ∞ and $H_P(S_\varphi)$ is the space of the functions with polynomial growth at 0 and ∞.

Obviously $H_0^\infty(S_\varphi) \subseteq H^\infty(S_\varphi) \subseteq H_P(S_\varphi) \subseteq H(S_\varphi)$ and $H^\infty(S_\varphi)$ is a Banach algebra if endowed with the norm $\|f\|_\infty = \sup_{z\in S_\varphi} |f(z)|$.

We suppose that A is a sectorial operator with spectral angle β, injective and with dense range.

If $f \in H_0^\infty(S_\varphi)$ (with $\varphi > \beta$) we define the bounded linear operator $f(A)$ through the integral

$$f(A) = \frac{1}{2\pi i} \int\limits_{\Gamma_\alpha} f(z)(zI - A)^{-1}\, dz$$

where $\alpha \in]\beta, \varphi[$ and Γ_α is the path in the complex plane composed by the two half-lines $\{\rho e^{\pm i\alpha}\colon \rho \in \mathbb{R}^+ \cup \{0\}\}$, oriented with decreasing imaginary part. Since the norm in $\mathcal{L}(X)$ of the integrand is summable, we have $f(A) \in \mathcal{L}(X)$. It is easy to show that this definition is independent of α.

We denote by ψ the function defined on $\mathbb{C} \setminus \{-1\}$ such that $\psi(z) = z/(1+z)^2$.

For every $f \in H_P(S_\varphi)$ there exists $m \in \mathbb{N}$ such that $\psi^m f \in H_0^\infty(S_\varphi)$; for such m we put

$$\mathcal{D}\big(f(A)\big) = \big\{ y \in X\colon \big(f\psi^m\big)(A)y \in \mathcal{D}\big(\big((I+A)^2A^{-1}\big)^m\big)\big\},$$
$$f(A) = \big((I+A)^2A^{-1}\big)^m \big(f\psi^m\big)(A).$$

The operator $f(A)$ is closed, since it is the composition of a bounded operator with a closed one and it is densely defined since $\mathcal{D}(((I+A)^2A^{-1})^m) = \mathcal{D}(A^m) \cap \mathcal{R}(A^m)$ is dense in X (see [12], Theorem 3.10). It is easy to show that this definition does not depend on m (see [12], Remark 4.2(a)). Moreover if f is a rational function then $f(A)$ coincides with the correspondent operator constructed through polynomials of A. In particular if $\mu \notin \overline{S}_\varphi$ and f is defined by $f(z) = (\mu - z)^{-1}$ then $f(A) = (\mu I - A)^{-1}$ (see [12], Theorem 5.5).

We say that A has bounded $H^\infty(S_\varphi)$ functional calculus if for every $f \in H^\infty(S_\varphi)$ the operator $f(A)$ is bounded and there exists $C \in \mathbb{R}^+$ (independent of f) such that $\|f(A)\| \leqslant C\|f\|_\infty$.

We shall need some facts concerning H^∞ functional calculus.

Proposition 3.1. *Let A be an injective sectorial operator with dense range. If there exists $C \in \mathbb{R}^+$ such that for $f \in H_0^\infty(S_\varphi)$ we have $\|f(A)\| \leqslant C\|f\|_\infty$, then A has bounded $H^\infty(S_\varphi)$ functional calculus.*

Proof. See [7], Corollary 2.2 or [12], Theorem 4.9. $\square$

Proposition 3.2. *Let A be an injective sectorial operator with dense range. If $f, g \in H_P(S_\varphi)$ then $f(A) + g(A) \subseteq (f + g)(A)$ and $f(A)g(A) \subseteq (fg)(A)$.*

If moreover $g(A) \in \mathcal{L}(X)$ then $f(A) + g(A) = (f + g)(A)$ and $f(A)g(A) = (fg)(A)$.

In particular if A has bounded $H^\infty(S_\varphi)$ functional calculus, then the operator $f \mapsto f(A)$ from $H^\infty(S_\varphi)$ to $\mathcal{L}(X)$ is an isomorphism of Banach algebras.

Proof. See [7] or [12], Theorem 4.5 and Corollary 4.6. $\square$

The equality $f(A)g(A) = (fg)(A)$ ensures that all the bounded operators defined through the functional calculus commute each other.

Proposition 3.3. *Let A be an injective sectorial operator with dense range. If $f \in H_P(S_\varphi)$ is such that $1/f \in H_P(S_\varphi)$ and $(1/f)(A) \in \mathcal{L}(X)$ then $f(A)$ is invertible with bounded inverse and $f(A)^{-1} = (1/f)(A)$.*

Proof. From Proposition 3.2 we have $f(A)(1/f)(A) = I$, and $(1/f)(A)f(A) \subseteq I$. The first equality implies that $(1/f)(A)$ is a right inverse of $f(A)$. Since $(1/f)(A)$ is bounded we have $\mathcal{D}((1/f)(A)f(A)) = \mathcal{D}(f(A))$ hence $(1/f)(A)$ is also a left inverse of $f(A)$. $\square$

If A is a sectorial operator with spectral angle β then it is possible to define the closed operator $A^{1/2}$ (see [16], Section 6.3 or [28], Section 1.15.1). It is known that $A^{1/2}$ is sectorial with spectral angle $\beta/2$ (see [16], Theorem 6.4.2) and $(A^{1/2})^2 = A$ (see [16], Eq. (6.3.28) or [28], Theorem 1.15.2(2)).

Proposition 3.4. *Let A be an injective sectorial operator with dense range. If A has bounded $H^\infty(S_\varphi)$ functional calculus then $A^{1/2}$ is an injective sectorial operator with dense range that has bounded $H^\infty(S_{\varphi/2})$ functional calculus.*

Proof. Let us put $B = A^{1/2}$. As already observed B is sectorial; moreover it is injective, since B^2 is injective, and has dense range, since its range contains the range of A.

Let $f \in H_0^\infty(S_{\varphi/2})$; for $w \in S_\varphi$ put $g(w) = f(w^{1/2})$; obviously $g \in H_0^\infty(S_\varphi)$. We have

$$
\begin{aligned}
g(A) &= \frac{1}{2\pi i} \int_{\Gamma_\alpha} g(w)(wI - A)^{-1}\, dw \\
&= \frac{1}{2\pi i} \int_{\Gamma_\alpha} f\bigl(w^{1/2}\bigr)\bigl(w^{1/2}I - B\bigr)^{-1}\bigl(w^{1/2}I + B\bigr)^{-1}\, dw \\
&= \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{f(w^{1/2})}{2w^{1/2}}\bigl(w^{1/2}I - B\bigr)^{-1}\, dw + \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{f(w^{1/2})}{2w^{1/2}}\bigl(w^{1/2}I + B\bigr)^{-1}\, dw.
\end{aligned}
$$

For every $w \in \mathbb{C} \setminus]-\infty, 0]$ we have $\operatorname{Re} w^{1/2} > 0$, therefore $-w^{1/2} \in \rho(B)$. Hence the function $w \mapsto (-w^{1/2}I - B)^{-1}$ is holomorphic in $\mathbb{C} \setminus]-\infty, 0]$ and the function

$$
w \mapsto \frac{f(w^{1/2})}{2w^{1/2}}\bigl(w^{1/2}I + B\bigr)^{-1} = -\frac{f(w^{1/2})}{2w^{1/2}}\bigl(-w^{1/2}I - B\bigr)^{-1}
$$

is holomorphic in S_φ, therefore $\int_{\Gamma_\alpha} \frac{f(w^{1/2})}{2w^{1/2}}(w^{1/2}I+B)^{-1}\,dw = 0$, hence

$$g(A) = \frac{1}{2\pi i}\int\limits_{\Gamma_\alpha} \frac{f(w^{1/2})}{2w^{1/2}}\big(w^{1/2}I - B\big)^{-1}\,dw = \frac{1}{2\pi i}\int\limits_{\Gamma_{\alpha/2}} f(z)(zI-B)^{-1}\,dz = f(B).$$

Since A has bounded $H^\infty(S_\varphi)$ functional calculus, then for a suitable $C \in \mathbb{R}^+$ (independent of f) we have

$$\big\|f(B)\big\| = \big\|g(A)\big\| \leqslant C\|g\|_\infty = C\|f\|_\infty.$$

This inequality holds for every $f \in H_0^\infty(S_{\varphi/2})$, hence by Proposition 3.1 B has bounded $H^\infty(S_{\varphi/2})$ functional calculus. □

Every sectorial operator with spectral angle less than $\pi/2$ is the opposite of the generator of a bounded analytic semigroup (see [15], Theorem II.4.6). When the operator is injective and has dense range, the semigroup can be obtained through the H^∞ functional calculus, as the following proposition shows.

Proposition 3.5. *Let A be an injective sectorial operator with dense range and spectral angle $\beta < \pi/2$. For $t \in \mathbb{R}^+ \cup \{0\}$ let $f_t \in H^\infty(S_{\pi/2})$ such that $f_t(z) = e^{-tz}$. Then $f_t(A)$ is bounded and $t \mapsto f_t(A)$ is the semigroup generated by $-A$.*

Proof. The analytic semigroup V generated by $-A$ is defined by

$$V(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-tz}(zI-A)^{-1}\,dz,$$

where, fixed $\alpha \in \,]\beta,\pi/2[$, Γ is the path in the complex plane composed by the two half-lines $\{\rho e^{\pm i\alpha}\colon\ \rho \in [1,\infty[\}$ and by the arc $\{e^{i\theta}\colon\ \theta \in [\alpha, 2\pi-\alpha]\}$, oriented from $\infty e^{i\alpha}$ to $\infty e^{-i\alpha}$ (see [15], Definition II.4.2). We have also

$$\begin{aligned} V(t) &= \frac{1}{2\pi i}\int\limits_{\Gamma}\big(e^{-tz}-(1+z)^{-1}\big)(zI-A)^{-1}\,dz + \frac{1}{2\pi i}\int\limits_{\Gamma}(1+z)^{-1}(zI-A)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_\alpha}\big(e^{-tz}-(1+z)^{-1}\big)(zI-A)^{-1}\,dz + (I+A)^{-1} \end{aligned}$$

by Cauchy theorem. Therefore, if g_t is the function defined by $g_t(z) = f_t(z) - 1/(1+z)$, we have $g_t \in H_0^\infty(S_{\pi/2})$ and $V(t) = g_t(A) + (I+A)^{-1}$. From Proposition 3.2 it follows that $V(t) = f_t(A)$. □

We end this section with a Convergence Lemma. Some kind of Convergence Lemma is proved in [26] and [7]; the version we need is similar to Proposition 2.10 of [19] or Proposition 5.1.4 of [20].

Proposition 3.6. *Let A be an injective sectorial operator with dense range. For $\delta \in]0,1[$ let $f_\delta \in H^\infty(S_\varphi)$ and $g \in H^\infty(S_\varphi)$. If f_δ converges to g pointwise as δ tends to 0, $\sup_{\delta\in]0,1[} \|f_\delta\|_\infty < \infty$ and $\sup_{\delta\in]0,1[} \|f_\delta(A)\| < \infty$ then $g(A)$ is bounded and $f_\delta(A)$ converges to $g(A)$ strongly as δ tends to 0.*

Proof. If $f_\delta, g \in H^\infty(S_\varphi)$ then $f_\delta\psi, g\psi \in H_0^\infty(S_\varphi)$ and by the dominated convergence theorem

$$\int_{\Gamma_\alpha} f_\delta(z)\psi(z)(zI-A)^{-1}\,dz \xrightarrow[\delta\to 0]{} \int_{\Gamma_\alpha} g(z)\psi(z)(zI-A)^{-1}\,dz$$

hence $(f_\delta\psi)(A)$ converges to $(g\psi)(A)$ in $\mathcal{L}(X)$. From this fact it follows that for every $y \in \mathcal{D}(A)\cap\mathcal{R}(A)$ $(f_\delta\psi)(A)(I+A)^2A^{-1}y$ converges to $(g\psi)(A)(I+A)^2A^{-1}y$, i.e. $f_\delta(A)y$ converges to $g(A)y$. The space $\mathcal{D}(A)\cap\mathcal{R}(A)$ is dense in X (see [12], Theorem 3.10), the operators $f_\delta(A)$ are uniformly bounded and $g(A)$ is closed, hence this convergence implies that $g(A)$ is bounded and it is easy to show that $f_\delta(A)y$ converges to $g(A)y$ for every $y \in X$. □

Obviously if A has bounded $H^\infty(S_\varphi)$ functional calculus, then the uniform boundedness of $f_\delta(A)$ is a consequence of the hypothesis on $\|f_\delta\|_\infty$.

4. Preliminary results

From now on we suppose that A is a closed, linear, densely defined operator in the Banach space X and satisfies the following hypotheses:

H1) $-A$ is sectorial with spectral angle $\beta \in]0,\pi[$;
H2) A is invertible with bounded inverse;
H3) there exists $\varphi \in]\beta,\pi[$ such that $-A$ has bounded $H^\infty(S_\varphi)$ functional calculus.

Example of operators that satisfy these hypotheses are certain elliptic operators (see [9]).

As observed in Section 3, if A satisfies H1 then it is possible to define the closed operator $(-A)^{1/2}$. We shall denote by B such operator.

From H2 it follows that B is injective and $B^{-1} = B(-A)^{-1}$ is closed and defined on the whole space, hence it is bounded.

From H3 and Proposition 3.4 it follows that B has bounded $H^\infty(S_{\varphi/2})$ functional calculus.

In the following we put $\theta = \varphi/2$. Note that $0 < \theta < \pi/2$.

We denote by V the analytic semigroup generated by $-B$.

Since B has bounded inverse it is easy to show that if $\omega \in \mathbb{R}^+$ is sufficiently small then $B-\omega$ is sectorial with spectral angle less than $\pi/2$, hence its opposite generates a bounded analytic semigroup. This semigroup is $t \mapsto e^{\omega t}V(t)$, hence there exist $M, \omega \in \mathbb{R}^+$ such that for $t \in \mathbb{R}^+$ we have $\|V(t)\| \leqslant Me^{-\omega t}$, that is the semigroup V has negative exponential type.

We shall make use of the real interpolation spaces $(X,Y)_{\alpha,p}$ between two Banach spaces X and Y (see [28], Section 1.3.2 for a definition). In particular, since the operator B is the infinitesimal generator of an analytic semigroup with negative exponential type, by [28], Theorem 1.14.5 and Theorem 1.15.2(f), $\|f\|_{(X,\mathcal{D}(B^2))_{j/2-1/(2p)},p}$ is equivalent to $\left(\int_0^T \|B^jV(x)f\|^p\,dx\right)^{1/p}$ with $T \in \mathbb{R}^+$ or $T = \infty$ and $j = 1,2$.

We require that the Banach space X has UMD property, or, equivalently, that the Hilbert transform is continuous from $L^p(\mathbb{R}; X)$ into itself ($1 < p < \infty$). It is known that Hilbert spaces, L^p spaces and ℓ^p spaces (with $1 < p < \infty$) have UMD property. For these results we refer to [6] and [27].

Proposition 4.1. *If X is a UMD Banach space and A satisfies hypotheses* H1, H2, H3*, then for all $f \in L^p(]a,b[; X)$ the function u such that*

$$u(x) = \int_a^x V(x-t) f(t)\,dt, \quad x \in]a,b[$$

is a solution of the Cauchy problem

$$\begin{cases} u'(x) + Bu(x) = f(x), & x \in]a,b[, \\ u(a) = 0 \end{cases}$$

belonging to the space $W^{1,p}(]a,b[; X) \cap L^p(]a,b[; \mathcal{D}(B))$ and there exists $C \in \mathbb{R}^+$ (*independent of f, a and b*) *such that*

$$\|u\|_{W^{1,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(B))} \leqslant C\|f\|_{L^p(]a,b[;X)}.$$

Moreover if $f \in L^p(]a,b[; \mathcal{D}(B))$ then $u \in W^{1,p}(]a,b[; \mathcal{D}(B)) \cap L^p(]a,b[; \mathcal{D}(B^2))$.

Proof. By Proposition 3.4 B has bounded $H^\infty(S_\theta)$ functional calculus, the function $z \mapsto e^{isz}$ belongs to $H^\infty(S_\theta)$ and $\sup_{z\in S_\theta} |e^{isz}| = e^{\theta|s|}$, therefore B has bounded imaginary powers and there exists C such that $\|B^{is}\| \leqslant Ce^{\theta|s|}$. Since X has UMD property, by [11] and Theorem 5.2 of [10], from this estimate it follows that for every $g \in L^p(]a,\infty[; \mathcal{D}(B))$ the problem

$$\begin{cases} v'(x) + Bv(x) = g(x), & x \in]a,\infty[, \\ v(a) = 0 \end{cases}$$

has one and only one solution v belonging to $W^{1,p}(]a,\infty[; X) \cap L^p(]a,\infty[; \mathcal{D}(B))$ and there exists $C \in \mathbb{R}^+$ (independent of g) such that

$$\|v\|_{W^{1,p}(]a,\infty[;X)} + \|v\|_{L^p(]a,\infty[;\mathcal{D}(B))} \leqslant C\|g\|_{L^p(]a,\infty[;X)}.$$

It can be proved that every solution v belonging to such space is the mild solution, i.e. $v(x) = \int_a^x V(x-t) g(t)\,dt$ (see [8], proof of Proposition 12.3).

This proves the first part of the proposition.

If $f \in L^p(]a,b[; \mathcal{D}(B))$ then the function $x \mapsto \int_a^x V(x-t) Bf(t)\,dt$ belongs to the space $L^p(]a,b[; \mathcal{D}(B))$, that is $Bu \in L^p(]a,b[; \mathcal{D}(B))$, hence $u \in L^p(]a,b[; \mathcal{D}(B^2))$. Moreover from the equality $u' = -Bu + f$ it follows that $u' \in L^p(]a,b[; \mathcal{D}(B))$, therefore $u \in W^{1,p}(]a,b[; \mathcal{D}(B))$. $\square$

Proposition 4.2. *For $g \in L^p(]a,b[;X)$ let $u_1, u_2 :]a,b[\to X$ defined by*

$$u_1(x) = \int_a^x V(x-t)B^{-1}g(t)\,dt, \qquad u_2(x) = \int_x^b V(t-x)B^{-1}g(t)\,dt.$$

Then the function $u = (u_1+u_2)/2$ belongs to $W^{2,p}(]a,b[;X) \cap L^p(]a,b[;\mathcal{D}(A))$ and is a solution of the equation

$$u''(x) + Au(x) = -g(x), \quad x \in]a,b[. \tag{NHE}$$

Moreover there exists $C \in \mathbb{R}^+$ (independent of g, a and b) such that

$$\|u\|_{W^{2,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(A))} \leqslant C\|g\|_{L^p(]a,b[;X)}.$$

Proof. Since $g \in L^p(]a,b[;X)$ we have $B^{-1}g \in L^p(]a,b[;\mathcal{D}(B))$, therefore, by Proposition 4.1, $u_1, u_2 \in W^{1,p}(]a,b[;\mathcal{D}(B)) \cap L^p(]a,b[;\mathcal{D}(B^2))$. For a.e. $x \in]a,b[$ we have

$$u_1'(x) = -Bu_1(x) + B^{-1}g(x), \qquad u_2'(x) = Bu_2(x) - B^{-1}g(x),$$

therefore

$$u_1'(x) + u_2'(x) = -Bu_1(x) + Bu_2(x).$$

Hence $u_1' + u_2' \in W^{1,p}(]a,b[;X)$, that is $u_1 + u_2 \in W^{2,p}(]a,b[;X)$, with

$$u_1''(x) + u_2''(x) = -Bu_1'(x) + Bu_2'(x) = B^2u_1(x) - g(x) + B^2u_2(x) - g(x).$$

Thus $u = (u_1+u_2)/2$ is a solution of Eq. (NHE).

From Proposition 4.1 it follows that there exists $C \in \mathbb{R}^+$ such that

$$\begin{aligned} \|u\|_{L^p(]a,b[;\mathcal{D}(A))} + \|u''\|_{L^p(]a,b[;X)} &\leqslant 2\|u\|_{L^p(]a,b[;\mathcal{D}(A))} + \|g\|_{L^p(]a,b[;X)} \\ &= 2\|Bu\|_{L^p(]a,b[;\mathcal{D}(B))} + \|g\|_{L^p(]a,b[;X)} \\ &\leqslant C\|g\|_{L^p(]a,b[;X)}. \qquad \square \end{aligned}$$

Proposition 4.3. *Let $c \in X$ and $u :]a,b[\to X$ be the function such that*

$$u(x) = V(x-a)c.$$

Then $u \in W^{2,p}(]a,b[;X) \cap L^p(]a,b[;\mathcal{D}(A))$ if and only if $c \in (X,\mathcal{D}(A))_{1-1/(2p),p}$ and in this case u is solution of the homogeneous equation

$$u''(x) + Au(x) = 0, \quad x \in]a,b[.$$

Moreover there exists $C \in \mathbb{R}^+$ (independent of c, a and b) such that

$$\|u\|_{W^{2,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(A))} \leqslant C\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}.$$

Proof. From [28], Theorem 1.14.5 it follows that $u \in L^p(]a,b[;\mathcal{D}(B^2))$ if and only if $c \in (X,\mathcal{D}(B^2))_{1-1/(2p),p} = (X,\mathcal{D}(A))_{1-1/(2p),p}$.

If $c \in (X,\mathcal{D}(A))_{1-1/(2p),p}$ then for a.e. $x \in]a,b[$ we have $u''(x) = B^2u(x)$, hence $u \in W^{2,p}(]a,b[;X)$. Moreover, as already noted, there exists $C \in \mathbb{R}^+$ such that

$$\left(\int_a^\infty \left\|B^2V(x-a)c\right\|^p dt\right)^{1/p} \leqslant C\|c\|_{(X,\mathcal{D}(B^2))_{1-1/(2p),p}},$$

therefore

$$\|u\|_{W^{2,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(A))} \leqslant C\left\|B^2u\right\|_{L^p(]a,b[;X)} \leqslant C\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}. \qquad \square$$

Proposition 4.4. *Let $g \in L^p(]a,b[;X)$. A function u is a solution of Eq.* (NHE) *belonging to $W^{2,p}(]a,b[;X) \cap L^p(]a,b[;\mathcal{D}(A))$ if and only if u is of the form*

$$u(x) = V(x-a)c + V(b-x)d + \frac{1}{2}\int_a^b V\big(|x-t|\big)B^{-1}g(t)\,dt$$

with $c,d \in (X,\mathcal{D}(A))_{1-1/(2p),p}$. Moreover there exists $C \in \mathbb{R}^+$ (independent of c, d, g, a and b) such that

$$\begin{aligned}&\|u\|_{W^{2,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(A))}\\ &\quad\leqslant C\big(\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|d\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|g\|_{L^p(]a,b[;X)}\big).\end{aligned}$$

Proof. The proposition follows easily from Propositions 4.2 and 4.3 (see [29], proof of Theorem 3 for a detailed proof). □

Proposition 4.5. *For every $f \in L^p(\mathbb{R}^+;\mathbb{R})$,*

$$\left(\int_{\mathbb{R}^+}\left|\int_{\mathbb{R}^+}\frac{f(s)}{t+s}\,ds\right|^p dt\right)^{1/p} \leqslant \frac{\pi}{\sin(\pi/p)}\|f\|_{L^p(\mathbb{R}^+;\mathbb{R})}.$$

Proof. See [21], Theorem 319. □

Proposition 4.6. *Let $g \in L^p(]a,b[;\mathcal{D}(B))$ and*

$$c = \int_a^b V(t-a)g(t)\,dt, \qquad d = \int_a^b V(b-t)g(t)\,dt.$$

Then $c, d \in (X, \mathcal{D}(A))_{1-1/(2p),p}$ *and there exists* $C \in \mathbb{R}^+$ *(independent of* a, b *and* g*) such that*

$$\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|d\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C\|g\|_{L^p(]a,b[;\mathcal{D}(B))}.$$

Proof. Since V is an analytic semigroup of negative exponential type, there exists $M \in \mathbb{R}^+$ such that $\|BV(x)\| \leqslant M/x$ for $x \in \mathbb{R}^+$ (see [15], Theorem II.4.6), hence, by Proposition 4.5, we have

$$\begin{aligned}\left(\int_0^\infty \left\|B^2V(x)c\right\|^p dx\right)^{1/p} &= \left(\int_0^\infty \left\|\int_a^b BV(x+t-a)Bg(t)\,dt\right\|^p dx\right)^{1/p} \\ &\leqslant \left(\int_0^\infty \left(\int_a^b \frac{M}{x+t-a}\left\|Bg(t)\right\|dt\right)^p dx\right)^{1/p} \\ &\leqslant \frac{\pi M}{\sin(\pi/p)}\left(\int_a^b \left\|Bg(t)\right\|^p dt\right)^{1/p},\end{aligned}$$

hence $c \in (X, \mathcal{D}(A))_{1-1/(2p),p}$ and $\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C\|g\|_{L^p(]a,b[;\mathcal{D}(B))}$.

The proof for d is analogous. □

Proposition 4.7. *For* k *positive integer and* $\alpha \in]0,1[$ *the space* $(X, \mathcal{D}(B^k))_{\alpha,p}$ *is invariant for the operator* $V(t)$ *and* $\sup_{t\in\mathbb{R}^+}\|V(t)\|_{\mathcal{L}((X,\mathcal{D}(B^k))_{\alpha,p})} < \infty$.

Proof. If $x \in \mathcal{D}(B)$ then $V(t)Bx = BV(t)x$, hence $V(t) \in \mathcal{L}(\mathcal{D}(B))$ and

$$\sup_{t\in\mathbb{R}^+}\left\|V(t)\right\|_{\mathcal{L}(\mathcal{D}(B))} \leqslant \sup_{t\in\mathbb{R}^+}\left\|V(t)\right\|_{\mathcal{L}(X)} < \infty;$$

the proposition follows easily by interpolation. □

Proposition 4.8. *For every positive integer* k *and* $\alpha \in]0, 1-1/k[$ *the operator* B^{-1} *is an isomorphism from* $(X, \mathcal{D}(B^k))_{\alpha,p}$ *onto* $(X, \mathcal{D}(B^k))_{\alpha+1/k,p}$.

Proof. See [28], Theorem 1.15.2(e). □

Proposition 4.9. *Let* $w, z \in \mathbb{C} \setminus \{0\}$. *We have*

$$|w+z| \geqslant \big(|w|+|z|\big)\left|\cos\frac{\arg w - \arg z}{2}\right|.$$

Proof. Let $\alpha = \arg w$ and $\beta = \arg z$. We have

$$|w+z|^2 = |w|^2\big(\cos^2\alpha + \sin^2\alpha\big) + 2|w||z|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + |z|^2\big(\cos^2\beta + \sin^2\beta\big)$$

$$= \left(|w|^2+|z|^2\right)\left(\cos^2\frac{\alpha-\beta}{2}+\sin^2\frac{\alpha-\beta}{2}\right)+2|w||z|\left(\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\alpha-\beta}{2}\right)$$

$$\geqslant \left(|w|^2+2|w||z|+|z|^2\right)\cos^2\frac{\alpha-\beta}{2}. \qquad \square$$

Proposition 4.10. *Let* $\alpha<\pi/2$ *and* $z\in S_\alpha$*.*

1. $|\arg(1-e^{-z})-\arg(1+e^{-z})|<\alpha$.
2. $|1+e^{-z}|\geqslant 1-\exp(-\pi/(2\tan\alpha))$.
3. $\frac{|z|\cos\alpha}{1+|z|\cos\alpha}\leqslant|1-e^{-z}|\leqslant\frac{2|z|}{1+|z|\cos\alpha}$.

Proof. (1) If $z\in S_\alpha$ then $|e^{-z}|<1$, hence $\operatorname{Re}(1-e^{-z})>0$ and $\operatorname{Re}(1+e^{-z})>0$, therefore

$$\arg\left(1-e^{-z}\right)-\arg\left(1+e^{-z}\right)=\arg\left(\frac{1-e^{-z}}{1+e^{-z}}\right).$$

We have

$$\frac{1-e^{-z}}{1+e^{-z}}=\frac{1-e^{-z}+e^{-\bar z}-e^{-z-\bar z}}{|1+e^{-z}|^2}=\frac{1+2ie^{-\operatorname{Re} z}\sin(\operatorname{Im} z)-e^{-2\operatorname{Re} z}}{|1+e^{-z}|^2},$$

hence

$$\arg\left(\frac{1-e^{-z}}{1+e^{-z}}\right)=\arctan\frac{2e^{-\operatorname{Re} z}\sin(\operatorname{Im} z)}{1-e^{-2\operatorname{Re} z}}=\arctan\frac{\sin(\operatorname{Im} z)}{\sinh(\operatorname{Re} z)}.$$

For $x\in\mathbb{R}^+\cup\{0\}$ we have $\sin x\leqslant x$ and $\sinh x\geqslant x$, hence

$$\left|\frac{\sin(\operatorname{Im} z)}{\sinh(\operatorname{Re} z)}\right|\leqslant\frac{|\operatorname{Im} z|}{\operatorname{Re} z}<\tan\alpha;$$

therefore

$$\left|\arg\left(1-e^{-z}\right)-\arg\left(1+e^{-z}\right)\right|=\left|\arctan\frac{\sin(\operatorname{Im} z)}{\sinh(\operatorname{Re} z)}\right|<\alpha.$$

(2) We have

$$\left|1+e^{-z}\right|^2=1+2e^{-\operatorname{Re} z}\cos(\operatorname{Im} z)+e^{-2\operatorname{Re} z}.$$

If $\operatorname{Re} z\leqslant\pi/(2\tan\alpha)$ then $|\operatorname{Im} z|\leqslant\pi/2$, therefore $\cos(\operatorname{Im} z)\geqslant 0$ and $|1+e^{-z}|^2\geqslant 1$, while if $\operatorname{Re} z\geqslant\pi/(2\tan\alpha)$ then

$$\left|1+e^{-z}\right|^2\geqslant 1-2e^{-\operatorname{Re} z}+e^{-2\operatorname{Re} z}=\left(1-e^{-\operatorname{Re} z}\right)^2\geqslant\left[1-\exp\left(-\pi/(2\tan\alpha)\right)\right]^2.$$

We can conclude that for all $z \in S_\alpha$ we have

$$\left|1 + e^{-z}\right| \geqslant 1 - \exp\bigl(-\pi/(2\tan\alpha)\bigr).$$

(3) For every $x \in \mathbb{R}^+$ we have $e^x \geqslant 1 + x$, hence $e^{-x} \leqslant 1/(1+x)$ and

$$1 - e^{-x} \geqslant 1 - \frac{1}{1+x} = \frac{x}{1+x}.$$

Therefore for $z \in S_\alpha$ we have

$$\left|1 - e^{-z}\right| \geqslant 1 - e^{-\operatorname{Re} z} \geqslant \frac{\operatorname{Re} z}{1 + \operatorname{Re} z},$$

but

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \leqslant \sqrt{(\operatorname{Re} z)^2 + \tan^2\alpha\,(\operatorname{Re} z)^2} = \frac{\operatorname{Re} z}{\cos\alpha},$$

so that

$$\left|1 - e^{-z}\right| \geqslant \frac{|z|\cos\alpha}{1 + |z|\cos\alpha}.$$

For every $x \in \mathbb{R}^+$ we have $e^{-x} \geqslant \max\{1 - x, 0\} \geqslant (1-x)/(1+x)$, hence

$$1 - e^{-x} \leqslant 1 - \frac{1-x}{1+x} = \frac{2x}{1+x}.$$

For $z \in S_\alpha$ we denote by Γ_z the segment in the complex plane that joins 0 to z,

$$\begin{aligned}\left|1 - e^{-z}\right| &= \left|\int_{\Gamma_z} e^{-w}\,dw\right| \leqslant \int_0^1 e^{-t\operatorname{Re} z}|z|\,dt \\ &= \bigl(1 - e^{-\operatorname{Re} z}\bigr)\frac{|z|}{\operatorname{Re} z} \leqslant \frac{2|z|}{1 + \operatorname{Re} z} \leqslant \frac{2|z|}{1 + |z|\cos\alpha}. \qquad \square\end{aligned}$$

5. Impedance and admittance operators

Let $g \in L^p(]a,b[;X)$, $\psi_a, \varkappa_b \in X$. We call L^p solution of the two-point mixed problem

$$\begin{cases} u''(x) + Au(x) = -g(x), & x \in]a,b[, \\ u(a) = \psi_a, \\ u'(b) = \varkappa_b, \end{cases} \tag{TPMP}$$

a function $u \in W^{2,p}(]a,b[;X) \cap L^p(]a,b[;\mathcal{D}(A))$, such that $u''(x) + Au(x) = -g(x)$, for a.e. $x \in]a,b[$ and $u(a) = \psi_a$, $u'(b) = \varkappa_b$.

Proposition 5.1. *Suppose X is a UMD Banach space and A is a closed operator in X satisfying hypotheses* H1, H2, H3. *For every $g \in L^p(]a,b[;X)$, $\psi_a \in (X,\mathcal{D}(A))_{1-1/(2p),p}$ and $\varkappa_b \in (X,\mathcal{D}(A))_{1/2-1/(2p),p}$ the two-point limit problem* (TPMP) *has one and only one L^p solution u. Moreover there exists $C \in \mathbb{R}^+$ (independent of g, ψ_a, $\varkappa_b$, a and b) such that*

$$\begin{aligned}&\|u\|_{W^{2,p}(]a,b[;X)} + \|u\|_{L^p(]a,b[;\mathcal{D}(A))}\\ &\quad \leqslant C\big(\|g\|_{L^p(]a,b[;X)} + \|\psi_a\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|\varkappa_b\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\big).\end{aligned}$$

By Proposition 4.4, an L^p solution of problem (TPMP) is a function of the form

$$u(x) = V(x-a)c + V(b-x)d + v(x), \tag{5.1}$$

with $c, d \in (X,\mathcal{D}(A))_{1-1/(2p),p}$ and $v :]a,b[\to X$ such that

$$v(x) = \frac{1}{2}\int_a^b V\big(|x-t|\big)B^{-1}g(t)\,dt.$$

Moreover the limit conditions $u(a) = \psi_a$ and $u'(b) = \varkappa_b$ must be satisfied.

The condition $u(a) = \psi_a$ becomes:

$$c + V(b-a)d + v(a) = \psi_a.$$

We have

$$u'(x) = -BV(x-a)c + BV(b-x)d + \frac{1}{2}\int_a^b \operatorname{sgn}(t-x)V\big(|x-t|\big)g(t)\,dt,$$

hence the condition $u'(b) = \varkappa_b$ becomes:

$$-BV(b-a)c + Bd - \frac{1}{2}\int_a^b V(b-t)g(t)\,dt = \varkappa_b.$$

Therefore we get the system:

$$\begin{cases} c + V(b-a)d = \psi_a - v(a),\\ -V(b-a)c + d = B^{-1}\varkappa_b + v(b).\end{cases}$$

From this it follows:

$$\begin{cases} \big(I + V(2b-2a)\big)c = \psi_a - v(a) - V(b-a)\big(B^{-1}\varkappa_b + v(b)\big),\\ \big(I + V(2b-2a)\big)d = V(b-a)\big(\psi_a - v(a)\big) + B^{-1}\varkappa_b + v(b).\end{cases} \tag{5.2}$$

In order to solve this system we have to invert the operator $I + V(2b-2a)$.

Lemma 5.2. *For every $t \in \mathbb{R}^+$ the operator $I + V(t)$ is invertible with bounded inverse and* $\sup_{t\in\mathbb{R}^+} \|(I + V(t))^{-1}\| < \infty$.

For every positive integer k and $\alpha \in]0, 1[$ the space $(X, \mathcal{D}(B^k))_{\alpha,p}$ is invariant for the operator $(I + V(t))^{-1}$ and $\sup_{t\in\mathbb{R}^+} \|(I + V(t))^{-1}\|_{\mathcal{L}((X,\mathcal{D}(B^k))_{\alpha,p})} < \infty$.

Proof. By Proposition 4.10, for $z \in S_\theta$ and $t \in \mathbb{R}^+$ we have

$$\left|1 + e^{-tz}\right| \geqslant 1 - \exp\bigl(-\pi/(2\tan\theta)\bigr) > 0,$$

hence the function $f_t : S_\theta \to \mathbb{C}$ such that $f_t(z) = 1 + e^{-tz}$ does not vanish on S_θ and the function $1/f_t$ belongs to $H^\infty(S_\theta)$, with norm bounded with respect to t. This allows us to conclude that the operator $I + V(t) = f_t(B)$ is invertible with bounded inverse $(I + V(t))^{-1} = (1/f_t)(B)$ and $\sup_{t\in\mathbb{R}^+} \|(I + V(t))^{-1}\| < \infty$.

It is easy to show that B^{-1} commutes with $V(t)$, hence for $x \in \mathcal{D}(B)$ we have $(I + V(t))^{-1}Bx = B(I + V(t))^{-1}x$, this fact implies that $(I + V(t))^{-1}$ belongs to $\mathcal{L}(\mathcal{D}(B))$ with norm estimated by the norm in $\mathcal{L}(\mathcal{D}(B))$. Hence we get the second part of the lemma by interpolation. □

By Lemma 5.2 the operator $I + V(2b - 2a)$ is invertible, hence system (5.2) has the only solution

$$c = \bigl(I + V(2b - 2a)\bigr)^{-1}\bigl[\psi_a - v(a) - V(b - a)\bigl(B^{-1}\varkappa_b + v(b)\bigr)\bigr],$$
$$d = \bigl(I + V(2b - 2a)\bigr)^{-1}\bigl[V(b - a)\bigl(\psi_a - v(a)\bigr) + B^{-1}\varkappa_b + v(b)\bigr].$$

We have $\psi_a \in (X, \mathcal{D}(A))_{1-1/(2p),p}$, $\varkappa_b \in (X, \mathcal{D}(A))_{1/(2p)-1/(2p),p}$, $B^{-1}g \in L^p(]a, b[; \mathcal{D}(B))$, hence from Propositions 4.6, 4.7, 4.8 and Lemma 5.2 we get $c, d \in (X, \mathcal{D}(A))_{1-1/(2p),p}$. The same propositions ensures that there exists $C \in \mathbb{R}^+$ such that

$$\begin{aligned}&\|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|d\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}\\ &\quad\leqslant C\bigl(\|g\|_{L^p(]a,b[;X)} + \|\psi_a\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|\varkappa_b\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\bigr).\end{aligned}$$

With this choice of c and d the function u defined in (5.1) is the only L^p solution of problem (TPMP) and the estimate stated in Proposition 5.1 holds.

This proof shows also that the unique solution of problem (TPMP) is the function u such that

$$\begin{aligned}u(x) &= V(x - a)\bigl(I + V(2b - 2a)\bigr)^{-1}\bigl[\psi_a - v(a) - V(b - a)\bigl(B^{-1}\varkappa_b + v(b)\bigr)\bigr]\\ &\quad+ V(b - x)\bigl(I + V(2b - 2a)\bigr)^{-1}\bigl[V(b - a)\bigl(\psi_a - v(a)\bigr) + B^{-1}\varkappa_b + v(b)\bigr] + v(x),\end{aligned}$$

that is

$$\begin{aligned}u(x) = \bigl(I + V(2b - 2a)\bigr)^{-1}\Bigl[&\bigl(V(x - a) + V(2b - a - x)\bigr)\psi_a\\ &+ \bigl(V(b - x) - V(x + b - 2a)\bigr)B^{-1}\varkappa_b\end{aligned}$$

$$- \frac{1}{2}\big(V(x-a) + V(2b-a-x)\big) \int_a^b V(t-a) B^{-1} g(t)\,dt$$

$$+ \frac{1}{2}\big(V(b-x) - V(x+b-2a)\big) \int_a^b V(b-t) B^{-1} g(t)\,dt \Bigg]$$

$$+ \frac{1}{2} \int_a^b V\big(|x-t|\big) B^{-1} g(t)\,dt \tag{5.3}$$

and we have

$$u'(x) = \big(I + V(2b-2a)\big)^{-1} \Bigg[\big(V(2b-a-x) - V(x-a)\big) B\psi_a$$

$$+ \big(V(b-x) + V(x+b-2a)\big)\varkappa_b$$

$$+ \frac{1}{2}\big(V(x-a) - V(2b-a-x)\big) \int_a^b V(t-a) g(t)\,dt$$

$$+ \frac{1}{2}\big(V(b-x) + V(x+b-2a)\big) \int_a^b V(b-t) g(t)\,dt \Bigg]$$

$$- \frac{1}{2} \int_a^b \operatorname{sgn}(x-t) V\big(|x-t|\big) g(t)\,dt. \tag{5.4}$$

If u is the solution of the problem (TPMP) we define the impedance operator for the interval $]a,b[$ at a as the operator such that

$$T_{a,b}(g, \psi_a, \varkappa_b) = u'(a)$$

and the admittance operator for the interval $]a,b[$ at b as the operator such that

$$U_{a,b}(g, \psi_a, \varkappa_b) = u(b).$$

From Eq. (5.4) we have

$$T_{a,b}(g, \psi_a, \varkappa_b) = T^*_{a,b} g + T^-_{a,b}\psi_a + T^+_{a,b}\varkappa_b,$$

with

$$T^*_{a,b} g = \big(I + V(2b-2a)\big)^{-1} \int_a^b \big(V(t-a) + V(2b-a-t)\big) g(t)\,dt,$$

$$T^-_{a,b}\psi_a = \big(I + V(2b-2a)\big)^{-1}\big(V(2b-2a) - I\big)B\psi_a,$$
$$T^+_{a,b}\varkappa_b = 2\big(I + V(2b-2a)\big)^{-1}V(b-a)\varkappa_b,$$

and from Eq. (5.3) we have

$$U_{a,b}(g, \psi_a, \varkappa_b) = U^*_{a,b}g + U^-_{a,b}\psi_a + U^+_{a,b}\varkappa_b,$$

with

$$U^*_{a,b}g = \big(I + V(2b-2a)\big)^{-1}\int_a^b \big(V(b-t) - V(b-2a+t)\big)B^{-1}g(t)\,dt,$$
$$U^-_{a,b}\psi_a = 2\big(I + V(2b-2a)\big)^{-1}V(b-a)\psi_a,$$
$$U^+_{a,b}\varkappa_b = \big(I + V(2b-2a)\big)^{-1}\big(I - V(2b-2a)\big)B^{-1}\varkappa_b.$$

Proposition 5.3. *The impedance operator $T_{a,b}$ is continuous from the space $L^p(]a,b[;X) \times (X, \mathcal{D}(A))_{1/2-1/(2p),p} \times (X, \mathcal{D}(A))_{1-1/(2p),p}$ into $(X, \mathcal{D}(A))_{1/2-1/(2p),p}$ and the admittance operator $U_{a,b}$ is continuous from the same space into $(X, \mathcal{D}(A))_{1-1/(2p),p}$. Moreover the norms of these operators are independent of a and b.*

Proof. The continuity of the operators follows immediately from the continuous dependence on the data of the solution of problem (TPMP) (Proposition 5.1) and the trace theorem for functions in $W^{2,p}(]a,b[;X) \cap L^p(]a,b[;\mathcal{D}(A))$ ([28], Theorem 1.14.5).

The fact that the estimates are independent of a and b follows easily from the uniform boundedness of the operators $(I + V(2b-2a))^{-1}$ between the interpolation spaces (Lemma 5.2), the boundedness of the semigroup V and Proposition 4.6. □

6. Resolution of the limit problem with impedance condition

By making use of the impedance operator, problem (ATP) is transformed in the following limit problem with impedance condition:

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[,\\ u_-(-1) = f_-, \\ p_-u_-'(0) = p_+T_{0,\delta}(g_+, u_-(0), f_+) + \chi. \end{cases} \tag{LPIC}$$

We call L^p solution of problem (LPIC) a function $u_- \in W^{2,p}(]-1,0[;X) \cap L^p(]-1,0[;\mathcal{D}(A))$, such that $u_-''(x) + Au_-(x) = -g_-(x)$ for a.e. $x \in]-1,0[$ and the limit conditions $u_-(-1) = f_-$ and $p_-u_-'(0) = p_+T_{0,\delta}(g_+, u_-(0), f_+) + \chi$ are satisfied.

Theorem 6.1. *Suppose X is a UMD Banach space and A is a closed operator in X satisfying hypotheses* H1, H2, H3. *For every $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_- \in (X, \mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$ and $\chi \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$ problem* (LPIC) *has one and only one L^p solution u_-. Moreover there exists $C \in \mathbb{R}^+$ (independent of g_-, g_+, f_-, f_+, χ, $\delta \in]0,1]$, $p_-, p_+ \in \mathbb{R}^+$) such that*

$$\begin{aligned}
&\|u_-\|_{W^{2,p}(]-1,0[;X)} + \|u_-\|_{L^p(]-1,0[;\mathcal{D}(A))} \\
&\quad \leqslant C\biggl(\|g_-\|_{L^p(]-1,0[;X)} + \frac{p_+}{(\delta p_+ + p_-)^{1/p}(p_+ + p_-)^{1/p'}}\|g_+\|_{L^p(]0,\delta[;X)} \\
&\qquad + \|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \frac{p_+}{\delta p_+ + p_-}\|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} \\
&\qquad + \frac{1}{\delta p_+ + p_-}\|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\biggr).
\end{aligned}$$

If u_- is solution of problem (LPIC) then it is solution of the mixed problem

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[, \\ u_-(-1) = f_-, \\ u_-'(0) = u_-'(0), \end{cases}$$

hence $u_-(0) = U_{-1,0}(g_-, f_-, u_-'(0))$ and the last condition in problem (LPIC) can be written as

$$\begin{aligned}
p_-u_-'(0) &= p_+\bigl(T^*_{0,\delta}g_+ + T^-_{0,\delta}u_-(0) + T^+_{0,\delta}f_+\bigr) + \chi \\
&= p_+\bigl(T^*_{0,\delta}g_+ + T^-_{0,\delta}\bigl(U^*_{-1,0}g_- + U^-_{-1,0}f_- + U^+_{-1,0}u_-'(0)\bigr) + T^+_{0,\delta}f_+\bigr) + \chi
\end{aligned}$$

that is

$$\bigl(p_- - p_+T^-_{0,\delta}U^+_{-1,0}\bigr)u_-'(0) = p_+\bigl(T^*_{0,\delta}g_+ + T^-_{0,\delta}U^*_{-1,0}g_- + T^-_{0,\delta}U^-_{-1,0}f_- + T^+_{0,\delta}f_+\bigr) + \chi.$$

Therefore if the operator $p_- - p_+T^-_{0,\delta}U^+_{-1,0}$ is invertible, then problem (LPIC) is equivalent to the following mixed problem

$$\begin{cases} u_-''(x) + Au_-(x) = -g_-(x), & x \in]-1,0[, \\ u_-(-1) = f_-, \\ u_-'(0) = \varkappa_0, \end{cases} \tag{6.1}$$

with

$$\varkappa_0 = \bigl(p_-I - p_+T^-_{0,\delta}U^+_{-1,0}\bigr)^{-1}\bigl[p_+\bigl(T^*_{0,\delta}g_+ + T^-_{0,\delta}U^*_{-1,0}g_- + T^-_{0,\delta}U^-_{-1,0}f_- + T^+_{0,\delta}f_+\bigr) + \chi\bigr].$$

We have

$$\begin{aligned}
p_-I - p_+T^-_{0,\delta}U^+_{-1,0} &= p_- - p_+\bigl(I + V(2\delta)\bigr)^{-1}\bigl(V(2\delta) - I\bigr)B\bigl(I + V(2)\bigr)^{-1}\bigl(I - V(2)\bigr)B^{-1} \\
&= \bigl(I + V(2\delta)\bigr)^{-1}\bigl(I + V(2)\bigr)^{-1} \\
&\quad \times \bigl[p_+\bigl(I - V(2\delta)\bigr)\bigl(I - V(2)\bigr) + p_-\bigl(I + V(2\delta)\bigr)\bigl(I + V(2)\bigr)\bigr];
\end{aligned}$$

hence we have to prove the invertibility of the operator

$$E_{\delta,p_-,p_+} = p_+\bigl(I - V(2\delta)\bigr)\bigl(I - V(2)\bigr) + p_-\bigl(I + V(2\delta)\bigr)\bigl(I + V(2)\bigr).$$

Lemma 6.2. *For $\delta > 0$, the operator*

$$E_{\delta,p_-,p_+} = p_+\bigl(I - V(2\delta)\bigr)\bigl(I - V(2)\bigr) + p_-\bigl(I + V(2\delta)\bigr)\bigl(I + V(2)\bigr)$$

is invertible with bounded inverse and there exists $C \in \mathbb{R}^+$ (independent of δ, p_- and p_+) such that

$$\bigl\| E^{-1}_{\delta,p_-,p_+} \bigr\| \leqslant C\frac{1}{p_-}.$$

Moreover for $\alpha \in]0,1[$ the operator E^{-1}_{δ,p_-,p_+} is bounded in $(X, \mathcal{D}(A))_{\alpha,p}$, with norm estimated in the same way as in $\mathcal{L}(X)$.

Proof. Let $e_{\delta,p_-,p_+} : S_\theta \to \mathbb{C}$ defined by

$$e_{\delta,p_-,p_+}(z) = p_+\bigl(1 - e^{-2\delta z}\bigr)\bigl(1 - e^{-2z}\bigr) + p_-\bigl(1 + e^{-2\delta z}\bigr)\bigl(1 + e^{-2z}\bigr).$$

The function e_{δ,p_-,p_+} is holomorphic and it is bounded since $\operatorname{Re} z > 0$ for $z \in S_\theta$; therefore $e_{\delta,p_-,p_+} \in H^\infty(S_\theta)$.

By Proposition 3.4 the operator B has bounded $H^\infty(S_\theta)$ functional calculus, hence from Propositions 3.5 and 3.2 we have $E_{\delta,p_-,p_+} = e_{\delta,p_-,p_+}(B)$.

From Proposition 4.9 it follows that for all $z \in S_\theta$ we have

$$\begin{aligned} \bigl|e_{\delta,p_-,p_+}(z)\bigr| \geqslant{}& \bigl(p_+\bigl|1 - e^{-2\delta z}\bigr|\bigl|1 - e^{-2z}\bigr| + p_-\bigl|1 + e^{-2\delta z}\bigr|\bigl|1 + e^{-2z}\bigr|\bigr) \\ & \times \left|\cos\frac{\arg(1 - e^{-2\delta z}) + \arg(1 - e^{-2z}) - \arg(1 + e^{-2\delta z}) - \arg(1 + e^{-2z})}{2}\right|. \end{aligned}$$

Since $2z, 2\delta z \in S_\theta$, by Proposition 4.10 we have

$$\begin{aligned} &\bigl|\arg\bigl(1 - e^{-2\delta z}\bigr) + \arg\bigl(1 - e^{-2z}\bigr) - \arg\bigl(1 + e^{-2\delta z}\bigr) - \arg\bigl(1 + e^{-2z}\bigr)\bigr| \\ &\quad \leqslant \bigl|\arg\bigl(1 - e^{-2z}\bigr) - \arg\bigl(1 + e^{-2z}\bigr)\bigr| + \bigl|\arg\bigl(1 - e^{-2\delta z}\bigr) - \arg\bigl(1 + e^{-2\delta z}\bigr)\bigr| < 2\theta, \end{aligned}$$

hence

$$\bigl|e_{\delta,p_-,p_+}(z)\bigr| \geqslant \bigl(p_+\bigl|1 - e^{-2\delta z}\bigr|\bigl|1 - e^{-2z}\bigr| + p_-\bigl|1 + e^{-2\delta z}\bigr|\bigl|1 + e^{-2z}\bigr|\bigr)\cos\theta \tag{6.2}$$

and from Proposition 4.10 we obtain

$$\bigl|e_{\delta,p_-,p_+}(z)\bigr| \geqslant p_-\bigl|1 + e^{-2\delta z}\bigr|\bigl|1 + e^{-2z}\bigr|\cos\theta \geqslant p_-\bigl[1 - \exp\bigl(-\pi/(2\tan\theta)\bigr)\bigr]^2\cos\theta.$$

Therefore the function e_{δ,p_-,p_+} does not vanish on S_θ and the function $1/e_{\delta,p_-,p_+}$ is bounded, hence it belongs to $H^\infty(S_\theta)$. Moreover $\|1/e_{\delta,p_-,p_+}\|_\infty \leqslant C/p_-$.

We can conclude that $E_{\delta,p_-,p_+} = e_{\delta,p_-,p_+}(B)$ is invertible with bounded inverse $E^{-1}_{\delta,p_-,p_+} = (1/e_{\delta,p_-,p_+})(B)$. The estimate of the norm of E^{-1}_{δ,p_-,p_+} follows from the estimate of $\|1/e_{\delta,p_-,p_+}\|_\infty$.

From the equality $E_{\delta,p_-,p_+}A^{-1} = A^{-1}E_{\delta,p_-,p_+}$, it follows $AE^{-1}_{\delta,p_-,p_+} = E^{-1}_{\delta,p_-,p_+}A$, hence E^{-1}_{δ,p_-,p_+} is a bounded operator in $\mathcal{D}(A)$. By interpolation we get the second part of the lemma. $\square$

The fact that the operator B is invertible allows us to improve the estimate of Lemma 6.2.

Lemma 6.3. *There exists $C \in \mathbb{R}^+$ such that for $\delta \in]0,1]$, $p_-, p_+ \in \mathbb{R}^+$,*

$$\left\| E^{-1}_{\delta,p_-,p_+} \right\| \leqslant C \frac{1}{\delta p_+ + p_-},$$

where the norm is in $\mathcal{L}(X)$ or in $\mathcal{L}((X,\mathcal{D}(A))_{\alpha,p})$.

Proof. Let $f_{\delta,p_-,p_+} : S_\theta \to \mathbb{C}$ defined by

$$f_{\delta,p_-,p_+}(z) = \left(1+z^{-1}\right)^2 e_{\delta,p_-,p_+}(z).$$

The function f_{δ,p_-,p_+} is holomorphic and, by (6.2) and Proposition 4.9, for every $z \in S_\theta$ we have

$$\begin{aligned} \left|f_{\delta,p_-,p_+}(z)\right| &= \left|1+z^{-1}\right|^2 \left|e_{\delta,p_-,p_+}(z)\right| \\ &\geqslant \left(1+|z|^{-1}\right)^2 \cos^2\frac{\theta}{2}\left(p_+\left|1-e^{-2\delta z}\right|\left|1-e^{-2z}\right| + p_-\left|1+e^{-2\delta z}\right|\left|1+e^{-2z}\right|\right)\cos\theta. \end{aligned} \tag{6.3}$$

From Proposition 4.10 we get

$$\left(1+|z|^{-1}\right)\left|1-e^{-2\delta z}\right| \geqslant \frac{1+|z|}{|z|}\frac{2\delta|z|\cos\theta}{1+2\delta|z|\cos\theta} \geqslant \delta\cos\theta, \tag{6.4}$$

$$\left(1+|z|^{-1}\right)\left|1-e^{-2z}\right| \geqslant \frac{1+|z|}{|z|}\frac{2|z|\cos\theta}{1+2|z|\cos\theta} \geqslant \cos\theta, \tag{6.5}$$

$$\left(1+|z|^{-1}\right)^2\left|1+e^{-2\delta z}\right|\left|1+e^{-2z}\right| \geqslant \left[1-\exp\left(-\pi/(2\tan\theta)\right)\right]^2, \tag{6.6}$$

hence there exists $C \in \mathbb{R}^+$ such that for $z \in S_\theta$ we have $|f_{\delta,p_-,p_+}(z)| \geqslant C(\delta p_+ + p_-)$. This proves that $f^{-1}_{\delta,p_-,p_+} \in H^\infty(S_\theta)$ and $\|f^{-1}_{\delta,p_-,p_+}\|_\infty \leqslant C/(\delta p_+ + p_-)$.

Taking into account the definition of f_{δ,p_-,p_+}, by Propositions 3.5 and 3.2, we have $E^{-1}_{\delta,p_-,p_+} = (I+B^{-1})^2 f^{-1}_{\delta,p_-,p_+}(B)$, hence in the space $\mathcal{L}(X)$ we have

$$\begin{aligned} \left\|E^{-1}_{\delta,p_-,p_+}\right\| &= \left\|\left(I+B^{-1}\right)^2 f^{-1}_{\delta,p_-,p_+}(B)\right\| \leqslant \left(1+\left\|B^{-1}\right\|\right)^2 \left\|f^{-1}_{\delta,p_-,p_+}(B)\right\| \\ &\leqslant C\left(1+\left\|B^{-1}\right\|\right)^2 \left\|f^{-1}_{\delta,p_-,p_+}\right\|_\infty \leqslant C\frac{1}{\delta p_+ + p_-}. \end{aligned}$$

The same argument of Lemma 6.2 allows to prove that this estimate holds in $\mathcal{L}((X,\mathcal{D}(A))_{\alpha,p})$. $\square$

Now we come back to problem (6.1). In order to apply Proposition 5.1 to this mixed problem we have to prove that $\varkappa_0 \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$.

By hypotheses $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_- \in (X, \mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$ and $\chi \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$, hence by Proposition 5.3

$$p_+\big(T^*_{0,\delta} g_+ + T^-_{0,\delta} U^*_{-1,0} g_- + T^-_{0,\delta} U^-_{-1,0} f_- + T^+_{0,\delta} f_+\big) + \chi \in \big(X, \mathcal{D}(A)\big)_{1/2-1/(2p),p};$$

by Lemma 6.2, this interpolation space is invariant for the operator E^{-1}_{δ,p_-,p_+}, therefore we have

$$\begin{aligned}
\varkappa_0 &= E^{-1}_{\delta,p_-,p_+}\big(I + V(2\delta)\big)\big(I + V(2)\big) \\
&\quad \times \big[p_+\big(T^*_{0,\delta} g_+ + T^-_{0,\delta} U^*_{-1,0} g_- + T^-_{0,\delta} U^-_{-1,0} f_- + T^+_{0,\delta} f_+\big) + \chi\big] \\
&\in \big(X, \mathcal{D}(A)\big)_{1/2-1/(2p),p}.
\end{aligned}$$

We can conclude that problem (6.1) has one and only one solution, hence the same is true for problem (LPIC). Moreover from Proposition 5.3 and Lemma 6.3 we get easily that

$$\begin{aligned}
\|\varkappa_0\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} \leqslant C \frac{1}{\delta p_+ + p_-} \big[& p_+\big(\|g_+\|_{L^p(]0,\delta[;X)} + \|g_-\|_{L^p(]-1,0[;X)} \\
& + \|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\big) \\
& + \|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\big],
\end{aligned}$$

hence from Proposition 5.1 we have

$$\begin{aligned}
&\|u_-\|_{W^{2,p}(]a,b[;X)} + \|u_-\|_{L^p(]a,b[;\mathcal{D}(A))} \\
&\quad \leqslant C \frac{1}{\delta p_+ + p_-} \big[(p_+ + p_-)\|g_-\|_{L^p(]-1,0[;X)} + p_+\|g_+\|_{L^p(]0,\delta[;X)} \\
&\qquad + (p_+ + p_-)\|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + p_+\|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} + \|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\big],
\end{aligned}$$

but the dependence on δ, p_- and p_+ in this estimate can be improved as stated in Theorem 6.1.

To this end we need an explicit expression of u_- and some more estimates concerning the operator E^{-1}_{δ,p_-,p_+}. We have

$$\begin{aligned}
\varkappa_0 &= \big(p_- I - p_+ T^-_{0,\delta} U^+_{-1,0}\big)^{-1}\big[p_+\big(T^*_{0,\delta} g_+ + T^-_{0,\delta} U^*_{-1,0} g_- + T^-_{0,\delta} U^-_{-1,0} f_- + T^+_{0,\delta} f_+\big) + \chi\big] \\
&= E^{-1}_{\delta,p_-,p_+}\big(I + V(2\delta)\big)\big(I + V(2)\big)\Bigg[p_+\big(I + V(2\delta)\big)^{-1}\Bigg[\int_0^\delta \big(V(t) + V(2\delta - t)\big) g_+(t)\,dt \\
&\quad + \big(V(2\delta) - I\big) B \big(I + V(2)\big)^{-1} \int_{-1}^0 \big(V(-t) - V(2+t)\big) B^{-1} g_-(t)\,dt \\
&\quad + 2\big(V(2\delta) - I\big) B \big(I + V(2)\big)^{-1} V(1) f_- + 2V(\delta) f_+\Bigg] + \chi\Bigg]
\end{aligned}$$

$$= E_{\delta,p_-,p_+}^{-1}\Bigg[p_+\big(I+V(2)\big)\int_0^{\delta}\big(V(t)+V(2\delta-t)\big)g_+(t)\,dt$$

$$+ p_+\big(V(2\delta)-I\big)\int_{-1}^{0}\big(V(-t)-V(2+t)\big)g_-(t)\,dt$$

$$+ 2p_+\big(V(2\delta)-I\big)BV(1)f_- + 2p_+\big(I+V(2\delta)\big)V(\delta)f_+ + \big(I+V(2\delta)\big)\big(I+V(2)\big)\chi\Bigg];$$

hence, recalling that u_- is solution of problem (6.1), that is of problem (TPMP) with $a=-1$, $b=0$, $g=g_-$, $\psi_a=f_-$ and $\varkappa_b=\varkappa_0$, from Eq. (5.3) we obtain

$$u_-(x) = \big(I+V(2)\big)^{-1}\Bigg[\big(V(x+1)+V(1-x)\big)f_-$$

$$+ \big(V(-x)-V(x+2)\big)B^{-1}E_{\delta,p_-,p_+}^{-1}\Bigg[p_+\big(I+V(2)\big)\int_0^{\delta}\big(V(t)+V(2\delta-t)\big)g_+(t)\,dt$$

$$+ p_+\big(V(2\delta)-I\big)\int_{-1}^{0}\big(V(-t)-V(2+t)\big)g_-(t)\,dt + 2p_+\big(V(2\delta)-I\big)BV(1)f_-$$

$$+ 2p_+\big(I+V(2\delta)\big)V(\delta)f_+ + \big(I+V(2\delta)\big)\big(I+V(2)\big)\chi\Bigg]$$

$$- \frac{1}{2}\big(V(x+1)+V(1-x)\big)\int_{-1}^{0}V(t+1)B^{-1}g_-(t)\,dt$$

$$+ \frac{1}{2}\big(V(-x)-V(x+2)\big)\int_{-1}^{0}V(-t)B^{-1}g_-(t)\,dt\Bigg] + \frac{1}{2}\int_{-1}^{0}V\big(|x-t|\big)B^{-1}g_-(t)\,dt;$$

therefore

$$u_-(x) = E_{\delta,p_-,p_+}^{-1}\Bigg[p_+\big(I-V(2\delta)\big)\big(V(x+1)-V(1-x)\big)f_-$$

$$+ p_-\big(I+V(2\delta)\big)\big(V(x+1)+V(1-x)\big)f_-$$

$$+ \big(V(-x)-V(x+2)\big)\big[2p_+V(\delta)B^{-1}f_+ + \big(I+V(2\delta)\big)B^{-1}\chi\big]$$

$$+ \frac{1}{2}p_+\big(I-V(2\delta)\big)\Bigg[\big(V(x+2)-V(-x)\big)\int_{-1}^{0}V(-t)B^{-1}g_-(t)\,dt$$

$$+ \big(V(1-x) - V(x+1)\big) \int_{-1}^{0} V(t+1)B^{-1}g_-(t)\,dt\Bigg]$$

$$+ \frac{1}{2}p_-\big(I + V(2\delta)\big)\Bigg[\big(V(-x) - V(x+2)\big) \int_{-1}^{0} V(-t)B^{-1}g_-(t)\,dt$$

$$- \big(V(1-x) + V(x+1)\big) \int_{-1}^{0} V(t+1)B^{-1}g_-(t)\,dt\Bigg]$$

$$+ p_+\big(V(-x) - V(x+2)\big) \int_{0}^{\delta} \big(V(t) + V(2\delta - t)\big)B^{-1}g_+(t)\,dt\Bigg]$$

$$+ \frac{1}{2}\int_{-1}^{0} V\big(|x-t|\big)B^{-1}g_-(t)\,dt. \tag{6.7}$$

Moreover we need the following lemmas.

Lemma 6.4. *There exists $C \in \mathbb{R}^+$ such that for $\delta \in \,]0,1]$, $p_-, p_+ \in \mathbb{R}^+$,*

$$\big\| p_+ E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)\big\| \leqslant C,$$

where the norm is in $\mathcal{L}(X)$ or in $\mathcal{L}((X, \mathcal{D}(A))_{\alpha,p})$.

Proof. First of all we note that $I - V(2)$ is invertible. Indeed for suitable $M, \omega \in \mathbb{R}^+$ we have $\|V(t)\| \leqslant Me^{-\omega t}$, therefore

$$\limsup_{n\to\infty}\big\|V(2)^n\big\|^{1/n} = \limsup_{n\to\infty}\big\|V(2n)\big\|^{1/n} \leqslant \lim_{n\to\infty}\big(Me^{-2n\omega}\big)^{1/n} = e^{-2\omega},$$

hence the spectral radius of $V(2)$ is less than 1. Therefore we have:

$$\begin{aligned} p_+ E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big) &= p_+ E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)\big(I - V(2)\big)\big(I - V(2)\big)^{-1} \\ &= E^{-1}_{\delta,p_-,p_+}\big[E_{\delta,p_-,p_+} - p_-\big(I + V(2\delta)\big)\big(I + V(2)\big)\big]\big(I - V(2)\big)^{-1} \\ &= \big[I - p_- E^{-1}_{\delta,p_-,p_+}\big(I + V(2\delta)\big)\big(I + V(2)\big)\big]\big(I - V(2)\big)^{-1} \end{aligned}$$

hence

$$\big\|p_+ E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)\big\| \leqslant \big(1 + p_-\big\|E^{-1}_{\delta,p_-,p_+}\big\|\big\|I + V(2\delta)\big\|\big\|I + V(2)\big\|\big)\big\|\big(I - V(2)\big)^{-1}\big\|$$

and this is bounded because of Lemma 6.2 and the boundedness of the semigroup V.

The same argument of Lemma 6.2 allows to prove that the estimate holds in $\mathcal{L}((X, \mathcal{D}(A))_{\alpha,p})$. $\square$

Since V is a bounded analytic semigroup generated by $-B$, there exists $C \in \mathbb{R}^+$ such that $\|BV(x)\| \leqslant C/x$ (see [15], Theorem II.4.6), hence $\|E^{-1}_{\delta,p_-,p_+}BV(x)\| \leqslant C/(x(\delta p_+ + p_-))$, but we need a better estimate.

Lemma 6.5. *There exists $C \in \mathbb{R}^+$ such that for $\delta \in]0,1]$, $p_-, p_+, x \in \mathbb{R}^+$ we have*

$$\left\|E^{-1}_{\delta,p_-,p_+}BV(x)\right\| \leqslant C\frac{x+\delta}{x(\delta(x+1)p_+ + (x+\delta)p_-)}.$$

Proof. Let $g_{\delta,p_-,p_+,x} : S_\theta \to \mathbb{C}$ defined by

$$g_{\delta,p_-,p_+,x}(z) = \frac{ze^{-xz}}{f_{\delta,p_-,p_+}(z)},$$

where f_{δ,p_-,p_+} is defined in the proof of Lemma 6.3.

The function $g_{\delta,p_-,p_+,x}$ is holomorphic, and it is the product of the bounded functions $z \mapsto ze^{-xz}$ and $1/f_{\delta,p_-,p_+}$, hence $g_{\delta,p_-,p_+,x} \in H^\infty(S_\theta)$. From Propositions 3.5 and 3.2 we have

$$E^{-1}_{\delta,p_-,p_+}BV(x) = \left(I + B^{-1}\right)^2 g_{\delta,p_-,p_+,x}(B),$$

therefore

$$\left\|E^{-1}_{\delta,p_-,p_+}BV(x)\right\| \leqslant \left\|\left(I + B^{-1}\right)^2\right\| \left\|g_{\delta,p_-,p_+,x}(B)\right\| \leqslant C\left\|g_{\delta,p_-,p_+,x}\right\|_\infty;$$

hence in order to prove the lemma it is sufficient to show that

$$\|g_{\delta,p_-,p_+,x}\|_\infty \leqslant C\frac{x+\delta}{x(\delta(x+1)p_+ + (x+\delta)p_-)}.$$

From Eq. (6.3) it follows that for $z \in S_\theta$,

$$\begin{aligned}\left|\frac{1}{g_{\delta,p_-,p_+,x}(z)}\right| &\geqslant \exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|z|}\\ &\quad\times \cos^2\frac{\theta}{2}\bigl(p_+\left|1-e^{-2\delta z}\right|\left|1-e^{-2z}\right| + p_-\left|1+e^{-2\delta z}\right|\left|1+e^{-2z}\right|\bigr)\cos\theta.\end{aligned}$$

From inequalities (6.4) and (6.5) we get

$$\begin{aligned}&\exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|z|}\left|1-e^{-2\delta z}\right|\left|1-e^{-2z}\right|\\ &\quad\geqslant \exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|)^2}{|z|^3}\frac{2\delta|z|\cos\theta}{1+2\delta|z|\cos\theta}\frac{2|z|\cos\theta}{1+2|z|\cos\theta}\\ &\quad\geqslant \frac{2\delta\cos^2\theta\,(1+x|z|\cos\theta)(1+|z|)}{|z|(1+2\delta|z|\cos\theta)},\end{aligned}$$

but the ratio of the values of two polynomial of second degree on $\mathbb{R}^+$ is greater than or equal to the minimum of the ratio of the corresponding coefficients of the two polynomial, therefore

$$\begin{aligned}\frac{(1+x|z|\cos\theta)(1+|z|)}{|z|(1+2\delta|z|\cos\theta)} &= \frac{x\cos\theta|z|^2+(1+x\cos\theta)|z|+1}{2\delta\cos\theta|z|^2+|z|}\\ &\geqslant \min\left\{\frac{x}{2\delta}, 1+x\cos\theta\right\}\\ &\geqslant \frac{x(1+x\cos\theta)}{2x+2\delta},\end{aligned}$$

hence

$$\exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|z|}\bigl|1-e^{-2\delta z}\bigr|\bigl|1-e^{-2z}\bigr| \geqslant \frac{\delta x\cos^2\theta(1+x\cos\theta)}{x+\delta}.$$

From inequality (6.6) we get

$$\begin{aligned}&\exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|z|}\bigl|1+e^{-2\delta z}\bigr|\bigl|1+e^{-2z}\bigr|\\ &\geqslant x\cos\theta\frac{\exp(x|z|\cos\theta)}{x|z|\cos\theta}\bigl[1-\exp\bigl(-\pi/(2\tan\theta)\bigr)\bigr]^2\\ &\geqslant x\cos\theta e\bigl[1-\exp\bigl(-\pi/(2\tan\theta)\bigr)\bigr]^2.\end{aligned}$$

Therefore there exists $C\in\mathbb{R}^+$ such that

$$\left|\frac{1}{g_{\delta,p_-,p_+,x}(z)}\right| \geqslant C\left(\frac{\delta x(1+x)}{\delta+x}p_+ + xp_-\right),$$

hence

$$\|g_{\delta,p_-,p_+,x}\|_\infty \leqslant C\frac{x+\delta}{x(\delta(1+x)p_+ + (x+\delta)p_-)}. \qquad \square$$

In order to complete the proof of Theorem 6.1 we have to prove the following estimates:

$$\left\|p_+E_{\delta,p_-,p_+}^{-1}\bigl(I-V(2\delta)\bigr)\int_{-1}^{0}V(-t)B^{-1}g_-(t)\,dt\right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C\|g_-\|_{L^p(]-1,0[;X)},$$

$$\left\|p_-E_{\delta,p_-,p_+}^{-1}\bigl(I+V(2\delta)\bigr)\int_{-1}^{0}V(-t)B^{-1}g_-(t)\,dt\right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C\|g_-\|_{L^p(]-1,0[;X)},$$

analogous estimates with $V(t+1)$ instead of $V(-t)$ and

$$
\begin{aligned}
&\left\| p_+ E^{-1}_{\delta,p_-,p_+} \int_0^\delta \big(V(t)+V(2\delta-t)\big)B^{-1}g_+(t)\,dt \right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \\
&\quad \leqslant C \frac{p_+}{(\delta p_+ + p_-)^{1/p}(p_+ + p_-)^{1/p'}} \|g_+\|_{L^p(]0,\delta[;X)}, \\
&\big\| p_+ E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big) f_- \big\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C \|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}, \\
&\big\| p_- E^{-1}_{\delta,p_-,p_+}\big(I + V(2\delta)\big) f_- \big\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C \|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}.
\end{aligned}
$$

The estimates involving g_- and f_- follow from the boundedness with respect to δ, p_- and p_+ of $\|p_+E^{-1}_{\delta,p_-,p_+}(I - V(2\delta))\|$ and $\|p_-E^{-1}_{\delta,p_-,p_+}(I + V(2\delta))\|$ (Lemmas 6.2 and 6.4) and the fact that

$$
\left\| \int_{-1}^0 V(-t)B^{-1}g_-(t)\,dt \right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \leqslant C\|g_-\|_{L^p(]-1,0[;X)}
$$

(see Proposition 4.6).

In order to prove the estimate involving g_+ note that

$$
\begin{aligned}
&\left\| p_+ E^{-1}_{\delta,p_-,p_+} \int_0^\delta \big(V(t)+V(2\delta-t)\big)B^{-1}g_+(t)\,dt \right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \\
&\leqslant C\left(\int_0^\infty \left\| B^2V(x)p_+E^{-1}_{\delta,p_-,p_+} \int_0^\delta \big(V(t)+V(2\delta-t)\big)B^{-1}g_+(t)\,dt \right\|^p dx \right)^{1/p} \\
&\leqslant C\left(\int_0^\infty \left(\int_0^\delta \big\|p_+E^{-1}_{\delta,p_-,p_+}BV(x+t)\big\| \big\|I+V(2\delta-2t)\big\| \, \big\|g_+(t)\big\|\,dt \right)^p dx \right)^{1/p}
\end{aligned}
$$

by Lemma 6.5

$$
\begin{aligned}
&\leqslant C\left(\int_0^\infty \left(\int_0^\delta \frac{(x+t+\delta)p_+}{(x+t)(\delta(x+t+1)p_+ + (x+t+\delta)p_-)} \big\|g_+(t)\big\|\,dt \right)^p dx \right)^{1/p} \\
&\leqslant C\left(\int_0^\infty \left(\int_0^\delta \left(\frac{p_+}{\delta(x+t+1)p_+ + (x+t+\delta)p_-} + \frac{p_+}{(x+t)(p_+ + p_-)} \right) \right.\right. \\
&\qquad \left.\left. \times \big\|g_+(t)\big\|\,dt \right)^p dx \right)^{1/p}
\end{aligned}
$$

by the Hölder inequality and Proposition 4.5

$$\leqslant C\left[\left(\int_0^\infty\left(\int_0^\delta\left(\frac{p_+}{\delta(x+t+1)p_+ + (x+t+\delta)p_-}\right)^{p'}dt\right)^{p/p'}dx\right)^{1/p}\right.$$
$$\left. + \frac{p_+}{p_+ + p_-}\right]\|g_+\|_{L^p(]0,\delta[;X)}.$$

We have

$$\int_0^\infty\left(\int_0^\delta\left(\frac{p_+}{\delta(x+t+1)p_+ + (x+t+\delta)p_-}\right)^{p'}dt\right)^{p/p'}dx$$
$$\leqslant \int_0^\infty \delta^{p/p'}\left(\frac{p_+}{\delta(x+1)p_+ + (x+\delta)p_-}\right)^{p}dx$$
$$= \delta^{p/p'}\frac{1}{p-1}\frac{p_+}{\delta p_+ + p_-}\left(\frac{p_+}{\delta p_+ + \delta p_-}\right)^{p-1}$$
$$= \frac{1}{p-1}\frac{p_+}{\delta p_+ + p_-}\left(\frac{p_+}{p_+ + p_-}\right)^{p-1},$$

hence

$$\left\|p_+ E^{-1}_{\delta,p_-,p_+}\int_0^\delta \big(V(t)+V(2\delta - t)\big)B^{-1}g_+(t)\,dt\right\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}$$
$$\leqslant C\left(\left(\frac{p_+}{\delta p_+ + p_-}\right)^{1/p}\left(\frac{p_+}{p_+ + p_-}\right)^{1/p'} + \frac{p_+}{p_+ + p_-}\right)$$
$$\leqslant C\frac{p_+}{(\delta p_+ + p_-)^{1/p}(p_+ + p_-)^{1/p'}}.$$

This concludes the proof of Theorem 6.1.

7. Estimates on the thin layer

In this section we complete our results on problem (ATP) with the study of the function u_+.

If u_- is the solution of (LPIC) (that depends on g_-, g_+, f_-, f_+ and χ) then u_+ is a solution of the following problem on the thin layer:

$$\begin{cases} u_+''(x) + Au_+(x) = -g_+(x), & x \in]0,\delta[, \\ u_+(0) = u_-(0), \\ u_+'(\delta) = f_+. \end{cases} \tag{TLP}$$

This is a particular case of problem (TPMP).

Theorem 7.1. *Suppose X is a UMD Banach space and A is a closed operator in X satisfying hypotheses* H1, H2, H3. *For every $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_- \in (X,\mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X,\mathcal{D}(A))_{1/2-1/(2p),p}$ and $\chi \in (X,\mathcal{D}(A))_{1/2-1/(2p),p}$ problem* (TLP) *has one and only one L^p solution u_+. Moreover there exists $C \in \mathbb{R}^+$* (*independent of g_-, g_+, f_-, f_+, χ, $\delta \in]0,1]$, $p_-, p_+ \in \mathbb{R}^+$*) *such that*

$$
\begin{aligned}
&\|u_+\|_{W^{2,p}(]0,\delta[;X)} + \|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))} \\
&\quad \leqslant C\biggl(\frac{p_-}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1/p}}\|g_-\|_{L^p(]-1,0[;X)} + \|g_+\|_{L^p(]0,\delta[;X)} \\
&\qquad + \frac{\delta^{1/p}p_-}{\delta p_+ + p_-}\|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + \left(\frac{p_+ + p_-}{\delta p_+ + p_-}\right)^{1/p'}\|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} \\
&\qquad + \frac{1}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1/p}}\|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\biggr).
\end{aligned}
$$

From the proof of Theorem 6.1 we know that $u_-(0) = U_{-1,0}(g_-, f_-, u'_-(0))$, hence, by Proposition 5.3, $u_-(0) \in (X,\mathcal{D}(A))_{1-1/(2p),p}$, therefore the existence and the uniqueness of the solution u_+ are an immediate consequence of Proposition 5.1. Moreover from Eq. (5.3) we get

$$
\begin{aligned}
u_+(x) = \bigl(I + V(2\delta)\bigr)^{-1}\biggl[&\bigl(V(x) + V(2\delta - x)\bigr)u_-(0) + \bigl(V(\delta - x) - V(x+\delta)\bigr)B^{-1}f_+ \\
&- \frac{1}{2}\bigl(V(x) + V(2\delta - x)\bigr)\int_0^\delta V(t)B^{-1}g_+(t)\,dt \\
&+ \frac{1}{2}\bigl(V(\delta - x) - V(x+\delta)\bigr)\int_0^\delta V(\delta - t)B^{-1}g_+(t)\,dt\biggr] \\
&+ \frac{1}{2}\int_0^\delta V\bigl(|x-t|\bigr)B^{-1}g_+(t)\,dt. \qquad (7.1)
\end{aligned}
$$

The function u_- is solution of problem (LPIC), hence from Eq. (6.7) we get

$$
\begin{aligned}
u_-(0) = E^{-1}_{\delta,p_-,p_+}\biggl[&2p_-\bigl(I + V(2\delta)\bigr)V(1)f_- \\
&+ \bigl(I - V(2)\bigr)\bigl[2p_+V(\delta)B^{-1}f_+ + \bigl(I + V(2\delta)\bigr)B^{-1}\chi\bigr] \\
&+ \frac{1}{2}\bigl[p_-\bigl(I + V(2\delta)\bigr) - p_+\bigl(I - V(2\delta)\bigr)\bigr]\bigl(I - V(2)\bigr)\int_{-1}^0 V(-t)B^{-1}g_-(t)\,dt \\
&- p_-\bigl(I + V(2\delta)\bigr)V(1)\int_{-1}^0 V(t+1)B^{-1}g_-(t)\,dt
\end{aligned}
$$

$$+ p_+\big(I - V(2)\big)\int_0^\delta \big(V(t) + V(2\delta - t)\big)B^{-1}g_+(t)\,dt\Bigg] + \frac{1}{2}\int_{-1}^0 V(-t)B^{-1}g_-(t)\,dt;$$

but

$$\begin{aligned}
&E_{\delta,p_-,p_+}^{-1}\big[p_-\big(I + V(2\delta)\big) - p_+\big(I - V(2\delta)\big)\big]\big(I - V(2)\big) + I \\
&\quad = E_{\delta,p_-,p_+}^{-1}\big[p_-\big(I + V(2\delta)\big)\big(I - V(2)\big) - p_+\big(I - V(2\delta)\big)\big(I - V(2)\big) \\
&\qquad + p_+\big(I - V(2\delta)\big)\big(I - V(2)\big) + p_-\big(I + V(2\delta)\big)\big(I + V(2)\big)\big] \\
&\quad = 2p_- E_{\delta,p_-,p_+}^{-1}\big(I + V(2\delta)\big),
\end{aligned}$$

hence we have

$$\begin{aligned}
u_-(0) = E_{\delta,p_-,p_+}^{-1}\Bigg[&2p_-\big(I + V(2\delta)\big)V(1)f_- \\
&+ \big(I - V(2)\big)\big[2p_+V(\delta)B^{-1}f_+ + \big(I + V(2\delta)\big)B^{-1}\chi\big] \\
&+ p_-\big(I + V(2\delta)\big)\int_{-1}^0 \big(V(-t) - V(t+2)\big)B^{-1}g_-(t)\,dt \\
&+ p_+\big(I - V(2)\big)\int_0^\delta \big(V(t) + V(2\delta - t)\big)B^{-1}g_+(t)\,dt\Bigg].
\end{aligned}$$

Therefore

$$\begin{aligned}
u_+(x) = E_{\delta,p_-,p_+}^{-1}\Bigg[&2p_-V(1)\big(V(x) + V(2\delta - x)\big)f_- \\
&+ \big(I + V(2\delta)\big)^{-1}\big[2p_+\big(I - V(2)\big)\big(V(x+\delta) + V(2\delta)V(\delta - x)\big) \\
&+ E_{\delta,p_-,p_+}\big(V(\delta - x) - V(x+\delta)\big)\big]B^{-1}f_+ + \big(I - V(2)\big)\big(V(x) + V(2\delta - x)\big)B^{-1}\chi \\
&+ p_-\big(V(x) + V(2\delta - x)\big)\int_{-1}^0 \big(V(-t) - V(t+2)\big)B^{-1}g_-(t)\,dt \\
&+ \big(I + V(2\delta)\big)^{-1}\bigg[p_+\big(I - V(2)\big) - \frac{1}{2}E_{\delta,p_-,p_+}\bigg] \\
&\times \big(V(x) + V(2\delta - x)\big)\int_0^\delta V(t)B^{-1}g_+(t)\,dt \\
&+ \big(I + V(2\delta)\big)^{-1}\bigg[p_+\big(I - V(2)\big)\big(V(x+\delta) + V(2\delta)V(\delta - x)\big)
\end{aligned}$$

$$+\frac{1}{2}E_{\delta,p_-,p_+}\bigl(V(\delta-x)-V(x+\delta)\bigr)\Bigr]\int_0^\delta V(\delta-t)B^{-1}g_+(t)\,dt\Bigr]$$

$$+\frac{1}{2}\int_0^\delta V\bigl(|x-t|\bigr)B^{-1}g_+(t)\,dt.$$

But

$$\begin{aligned}
&2p_+\bigl(I-V(2)\bigr)-E_{\delta,p_-,p_+}\\
&\quad=2p_+\bigl(I-V(2)\bigr)-p_+\bigl(I-V(2\delta)\bigr)\bigl(I-V(2)\bigr)-p_-\bigl(I+V(2\delta)\bigr)\bigl(I+V(2)\bigr)\\
&\quad=\bigl(I+V(2\delta)\bigr)\bigl[p_+\bigl(I-V(2)\bigr)-p_-\bigl(I+V(2)\bigr)\bigr]
\end{aligned}$$

and

$$\begin{aligned}
&2p_+\bigl(I-V(2)\bigr)V(2\delta)+E_{\delta,p_-,p_+}\\
&\quad=2p_+\bigl(I-V(2)\bigr)V(2\delta)+p_+\bigl(I-V(2\delta)\bigr)\bigl(I-V(2)\bigr)+p_-\bigl(I+V(2\delta)\bigr)\bigl(I+V(2)\bigr)\\
&\quad=\bigl(I+V(2\delta)\bigr)\bigl[p_+\bigl(I-V(2)\bigr)+p_-\bigl(I+V(2)\bigr)\bigr],
\end{aligned}$$

hence

$$\begin{aligned}
u_+(x)=E_{\delta,p_-,p_+}^{-1}\Bigl[&2p_-V(1)\bigl(V(x)+V(2\delta-x)\bigr)f_-\\
&+p_+\bigl(I-V(2)\bigr)\bigl(V(\delta-x)+V(x+\delta)\bigr)B^{-1}f_+\\
&+p_-\bigl(I+V(2)\bigr)\bigl(V(\delta-x)-V(x+\delta)\bigr)B^{-1}f_+\\
&+\bigl(I-V(2)\bigr)\bigl(V(x)+V(2\delta-x)\bigr)B^{-1}\chi\\
&+p_-\bigl(V(x)+V(2\delta-x)\bigr)\int_{-1}^0\bigl(V(-t)-V(t+2)\bigr)B^{-1}g_-(t)\,dt\\
&+\frac{1}{2}p_+\bigl(I-V(2)\bigr)\Bigl[\bigl(V(\delta-x)+V(x+\delta)\bigr)\int_0^\delta V(\delta-t)B^{-1}g_+(t)\,dt\\
&+\bigl(V(x)+V(2\delta-x)\bigr)\int_0^\delta V(t)B^{-1}g_+(t)\,dt\Bigr]\\
&+\frac{1}{2}p_-\bigl(I+V(2)\bigr)\Bigl[\bigl(V(\delta-x)-V(x+\delta)\bigr)\int_0^\delta V(\delta-t)B^{-1}g_+(t)\,dt
\end{aligned}$$

$$- \big(V(x) + V(2\delta - x)\big) \int_0^\delta V(t) B^{-1} g_+(t)\, dt \Big]\Big]$$

$$+ \frac{1}{2} \int_0^\delta V\big(|x-t|\big) B^{-1} g_+(t)\, dt. \tag{7.2}$$

From Propositions 4.3, 4.6, 4.7, 4.8 and 6.3 we obtain easily that there exists $C \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\|u_+\|_{W^{2,p}(]a,b[;X)} + \|u_+\|_{L^p(]a,b[;\mathcal{D}(A))} \\
&\quad \leqslant C \frac{1}{\delta p_+ + p_-} \big[p_- \|g_-\|_{L^p(]-1,0[;X)} + (p_+ + p_-)\|g_+\|_{L^p(]0,\delta[;X)} + p_- \|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} \\
&\qquad + (p_+ + p_-)\|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} + \|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} \big].
\end{aligned}$$

In order to obtain the best dependence on δ, p_- and p_+, stated in Theorem 7.1, we need a lemma that gives an estimate of the norm of the function $x \mapsto V(x)c$ on $]0,\delta[$ in term of the norm of c in an interpolation space, i.e. we need to improve the estimate of Proposition 4.3 when the length of the interval tends to 0.

Lemma 7.2. *Let $c \in (X, \mathcal{D}(A))_{1-1/(2p),p}$, $G \in \mathcal{L}(X)$ that commutes with A and $u :]0,\delta[\to X$ be the function such that*

$$u(x) = GV(x)c.$$

Then there exists $C \in \mathbb{R}^+$ (independent of c and δ) such that

$$\begin{aligned}
&\|u\|_{W^{2,p}(]0,\delta[;X)} + \|u\|_{L^p(]0,\delta[;\mathcal{D}(A))} \\
&\quad \leqslant C \left(\sum_{k=0}^{\infty} \big\| G\big(I - V(2\delta)\big) V(k\delta) \big\|^{p'} \right)^{1/p'} \|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}.
\end{aligned}$$

Proof. We have $u'' = -Au$, hence it is sufficient to estimate $\|u\|_{L^p(]0,\delta[;\mathcal{D}(A))}$.

Since in general

$$\|\phi\|_{L^p(]0,\delta[;\mathbb{R})} = \sup_{\substack{\psi \in L^{p'}(]0,\delta[;\mathbb{R}) \\ \|\psi\|=1}} \int_0^\delta \big|\psi(x)\phi(x)\big|\, dx,$$

it is sufficient to estimate $\int_0^\delta |\psi(x)| \|Au(x)\|\, dx$ independently of $\psi \in L^{p'}(]0,\delta[;\mathbb{R})$ with $\|\psi\| = 1$. We have

$$\begin{aligned}
&\int_0^\delta |\psi(x)| \|GAV(x)c\|\,dx \\
&\quad\leqslant \int_0^\delta |\psi(x)| \left\| G\big(I - V(2\delta)\big)\sum_{k=0}^\infty V(2\delta)^k AV(x)c \right\| dx \\
&\quad\leqslant \sum_{k=0}^\infty \int_0^\delta |\psi(x)| \big\| G\big(I - V(2\delta)\big)V(k\delta)\big\| \|AV(x+k\delta)c\|\,dx \\
&\quad\leqslant \left(\sum_{k=0}^\infty \int_0^\delta |\psi(x)|^{p'} \big\| G\big(I - V(2\delta)\big)V(k\delta)\big\|^{p'} dx\right)^{1/p'} \left(\sum_{k=0}^\infty \int_0^\delta \|AV(x+k\delta)c\|^p\,dx\right)^{1/p} \\
&\quad\leqslant \left(\sum_{k=0}^\infty \big\| G\big(I - V(2\delta)\big)V(k\delta)\big\|^{p'}\right)^{1/p'} \left(\sum_{k=0}^\infty \int_{k\delta}^{k\delta+\delta} \|AV(x)c\|^p\,dx\right)^{1/p} \\
&\quad\leqslant C\left(\sum_{k=0}^\infty \big\| G\big(I - V(2\delta)\big)V(k\delta)\big\|^{p'}\right)^{1/p'} \|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}. \qquad \square
\end{aligned}$$

We shall apply Lemma 7.2 with $G = E^{-1}_{\delta,p_-,p_+}$, hence we need the following estimate.

Lemma 7.3. *There exists $C \in \mathbb{R}^+$ such that for $\delta \in]0,1]$, $p_-, p_+, x \in \mathbb{R}^+$ we have*

$$\big\| E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)V(x)\big\| \leqslant C\frac{\delta}{\delta(x+1)p_+ + (x+\delta)p_-}.$$

Proof. Let $h_{\delta,p_-,p_+} : S_\theta \to \mathbb{C}$ defined by

$$h_{\delta,p_-,p_+}(z) = \frac{(1-e^{-2\delta z})e^{-xz}}{f_{\delta,p_-,p_+}(z)},$$

where f_{δ,p_-,p_+} is defined in the proof of Lemma 6.3.

The function h_{δ,p_-,p_+} is the product of the bounded holomorphic functions $z \mapsto (1-e^{-2\delta z})e^{-xz}$ and $1/f_{\delta,p_-,p_+}$, hence $h_{\delta,p_-,p_+} \in H^\infty(S_\theta)$. From Propositions 3.5 and 3.2 we have

$$E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)V(x) = \big(I + B^{-1}\big)^2 h_{\delta,p_-,p_+}(B),$$

therefore

$$\big\| E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)V(x)\big\| \leqslant \big\|\big(I + B^{-1}\big)^2\big\| \big\|h_{\delta,p_-,p_+}(B)\big\| \leqslant C\|h_{\delta,p_-,p_+}\|_\infty;$$

hence in order to prove the lemma it is sufficient to show that

$$\|h_{\delta,p_-,p_+}\|_\infty \leqslant C\frac{\delta}{\delta(x+1)p_+ + (x+\delta)p_-}.$$

From Eq. (6.3) it follows that for $z \in S_\theta$

$$\begin{aligned}\left|\frac{1}{h_{\delta,p_-,p_+}(z)}\right| &\geqslant \exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|1-e^{-2\delta z}|}\\ &\quad\times \cos^2\frac{\theta}{2}\bigl(p_+\bigl|1-e^{-2\delta z}\bigr|\bigl|1-e^{-2z}\bigr| + p_-\bigl|1+e^{-2\delta z}\bigr|\bigl|1+e^{-2z}\bigr|\bigr)\cos\theta.\end{aligned}$$

From inequalities (6.4) and (6.5) we get

$$\begin{aligned}&\exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|1-e^{-2\delta z}|}\bigl|1-e^{-2\delta z}\bigr|\bigl|1-e^{-2z}\bigr|\\ &\quad\geqslant \exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|)^2}{|z|^2}\frac{2|z|\cos\theta}{1+2|z|\cos\theta}\\ &\quad\geqslant \frac{\cos\theta(1+x|z|\cos\theta)(1+|z|)}{|z|}\\ &\quad= \cos\theta\bigl(|z| + x\cos\theta + 1 + |z|^{-1}\bigr)\\ &\quad\geqslant \cos\theta(x\cos\theta+3).\end{aligned}$$

From inequality (6.6) and Proposition 4.10 we get

$$\begin{aligned}&\exp\bigl(x|z|\cos\theta\bigr)\frac{(1+|z|^{-1})^2}{|1-e^{-2\delta z}|}\bigl|1+e^{-2\delta z}\bigr|\bigl|1+e^{-2z}\bigr|\\ &\quad\geqslant \bigl(1+x|z|\cos\theta\bigr)\frac{1+2\delta|z|\cos\theta}{4\delta|z|}\bigl[1-\exp\bigl(-\pi/(2\tan\theta)\bigr)\bigr]^2\\ &\quad\geqslant \frac{(x+2\delta)\cos\theta}{4\delta}\bigl[1-\exp\bigl(-\pi/(2\tan\theta)\bigr)\bigr]^2.\end{aligned}$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\left|\frac{1}{h_{\delta,p_-,p_+}(z)}\right| \geqslant C\left((x+1)p_+ + \frac{x+\delta}{\delta}p_-\right),$$

hence

$$\|h_{\delta,p_-,p_+}\|_\infty \leqslant C\frac{\delta}{\delta(x+1)p_+ + (x+\delta)p_-}. \qquad \square$$

From Lemmas 7.2 and 7.3 we get the following estimate.

Lemma 7.4. *Let $c \in (X, \mathcal{D}(A))_{1-1/(2p),p}$ and $u :]0,\delta[\to X$ be the function such that*

$$u(x) = E^{-1}_{\delta,p_-,p_+} V(x)c.$$

Then there exists $C \in \mathbb{R}^+$ (independent of c, δ, p_- and p_+) such that

$$\|u\|_{W^{2,p}(]0,\delta[;X)} + \|u\|_{L^p(]0,\delta[;\mathcal{D}(A))} \leqslant C \frac{1}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1/p}} \|c\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}}.$$

Proof. From Lemma 7.3 we have

$$\left\| E^{-1}_{\delta,p_-,p_+}\big(I - V(2\delta)\big)V(k\delta)\right\| \leqslant C \frac{\delta}{\delta(k\delta+1)p_+ + (k\delta+\delta)p_-}$$

and

$$\begin{aligned}
&\left(\sum_{k=0}^{\infty}\left(\frac{1}{(k\delta+1)p_+ + (k+1)p_-}\right)^{p'}\right)^{1/p'} \\
&\quad\leqslant \left(\frac{1}{p_+ + p_-} + \int_0^{\infty}\left(\frac{1}{(t\delta+1)p_+ + (t+1)p_-}\right)^{p'} dt\right)^{1/p'} \\
&\quad= \left(\frac{1}{p_+ + p_-} + \frac{1}{p'-1}\frac{1}{\delta p_+ + p_-}\left(\frac{1}{p_+ + p_-}\right)^{p'-1}\right)^{1/p'} \\
&\quad\leqslant C\left(\frac{1}{\delta p_+ + p_-}\left(\frac{1}{p_+ + p_-}\right)^{p'-1}\right)^{1/p'} \\
&\quad= C\frac{1}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1/p}},
\end{aligned}$$

hence the lemma follows immediately from Lemma 7.2. □

From this lemma we get

$$\begin{aligned}
&\|u_+\|_{W^{2,p}(]a,b[;X)} + \|u_+\|_{L^p(]a,b[;\mathcal{D}(A))} \\
&\quad\leqslant C\frac{1}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1/p}}\big[p_-\|g_-\|_{L^p(]-1,0[;X)} + (p_+ + p_-)\|g_+\|_{L^p(]0,\delta[;X)} \\
&\qquad + p_-\|f_-\|_{(X,\mathcal{D}(A))_{1-1/(2p),p}} + (p_+ + p_-)\|f_+\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}} + \|\chi\|_{(X,\mathcal{D}(A))_{1/2-1/(2p),p}}\big].
\end{aligned}$$

The dependence on g_+ and f_- in this estimate can be improved. In fact we have

$$\left\| AE^{-1}_{\delta,p_-,p_+}V(\cdot)\int_0^\delta V(t)B^{-1}g_+(t)\,dt\right\|_{L^p(]0,\delta[;X)}$$
$$\leqslant \left(\int_0^\delta\left(\int_0^\delta \left\|E^{-1}_{\delta,p_-,p_+}BV(x+t)\right\|\left\|g_+(t)\right\|dt\right)^p dx\right)^{1/p}$$

by Lemma 6.5

$$\leqslant C\left(\int_0^\delta\left(\int_0^\delta \frac{x+t+\delta}{(x+t)(\delta(x+t+1)p_+ + (x+t+\delta)p_-)}\left\|g_+(t)\right\|dt\right)^p dx\right)^{1/p}$$
$$\leqslant C\left(\int_0^\delta\left(\int_0^\delta\left(\frac{1}{\delta(x+t+1)p_+ + (x+t+\delta)p_-}+\frac{1}{(x+t)(p_+ + p_-)}\right)\right.\right.$$
$$\left.\left.\times\left\|g_+(t)\right\|dt\right)^p dx\right)^{1/p}$$

by the Hölder inequality and Proposition 4.5

$$\leqslant C\left[\left(\int_0^\delta\left(\int_0^\delta\left(\frac{1}{\delta(x+t+1)p_+ + (x+t+\delta)p_-}\right)^{p'}dt\right)^{p/p'}dx\right)^{1/p}\right.$$
$$\left.+\frac{1}{p_+ + p_-}\right]\|g_+\|_{L^p(]0,\delta[;X)}.$$

But

$$\left(\int_0^\delta\left(\int_0^\delta\left(\frac{1}{\delta(x+t+1)p_+ + (x+t+\delta)p_-}\right)^{p'}dt\right)^{p/p'}dx\right)^{1/p}$$
$$\leqslant\left(\int_0^\delta\left(\int_0^\delta\left(\frac{1}{\delta p_+ + \delta p_-}\right)^{p'}dt\right)^{p/p'}dx\right)^{1/p}=\frac{1}{p_+ + p_-},$$

hence

$$\left\|x\mapsto AE^{-1}_{\delta,p_-,p_+}V(x)\int_0^\delta V(t)B^{-1}g_+(t)\,dt\right\|_{L^p(]0,\delta[;X)}\leqslant C\frac{1}{p_+ + p_-}\|g_+\|_{L^p(]0,\delta[;X)}.$$

From this inequality it follows immediately that the norm in $L^p(]0,\delta[;\mathcal{D}(A))$ of each term containing g_+ in the representation of u_+ in formula (7.2) is less than or equal to $C\|g_+\|$, with C independent of δ, p_- and p_+. Because of the connection between u_+'' and Au_+, the same estimate is true for the norm in $W^{2,p}(]0,\delta[;X)$.

Moreover

$$\begin{aligned}&\left\|AE_{\delta,p_-,p_+}^{-1}p_-V(1)\big(V(\cdot)+V(2\delta-\cdot)\big)f_-\right\|_{L^p(]0,\delta[;X)}\\&\quad\leqslant\left(\int_0^\delta\left\|E_{\delta,p_-,p_+}^{-1}p_-V(1)\big(V(x)+V(2\delta-x)\big)B^2f_-\right\|^p dx\right)^{1/p}\end{aligned}$$

by Lemma 6.3

$$\begin{aligned}&\leqslant C\left(\int_0^\delta\left(\frac{p_-}{\delta p_++p_-}\|f_-\|\right)^p dx\right)^{1/p}\\&=C\frac{\delta^{1/p}p_-}{\delta p_++p_-}\|f_-\|,\end{aligned}$$

hence the norm in $L^p(]0,\delta[;\mathcal{D}(A))$ of the term containing f_- of the expression of u_+ in Eq. (7.2) is less than or equal to $C\delta^{1/p}p_-/(\delta p_++p_-)$, with C independent of δ, p_- and p_+ and this is true also for the norm in $W^{2,p}(]0,\delta[;X)$.

This concludes the proof of Theorem 7.1.

8. Estimates from below

In this section we show that the dependence on δ, p_- and p_+ in the estimates of Theorems 6.1 and 7.1 cannot be improved.

In the following (u_-,u_+) will always denote the L^p solution of problem (ATP). We suppose $\delta\leqslant 1$.

We'll make use of the fact that, because of the concavity of the function $y\mapsto 1-e^{-y}$, for every $y\in\mathbb{R}^+$ we have $1-e^{-y}\leqslant y$ and from Proposition 4.10 we have

$$1-e^{-y}\geqslant\frac{y}{1+y}.$$

From this it follows that

$$e_{\delta,p_-,p_+}(1)=p_+\big(1-e^{-2\delta}\big)\big(1-e^{-2}\big)+p_-\big(1+e^{-2\delta}\big)\big(1+e^{-2}\big)\leqslant 4(\delta p_++p_-).$$

In Examples 8.1–8.5 we have $X=\mathbb{C}$ and A is the opposite of the identity operator.

The space $\mathbb{C}$ has UMD property, A satisfies the hypotheses H1, H2, H3, the operator $B=(-A)^{1/2}$ is the identity operator and V is such that $V(x)u=e^{-x}u$. Moreover we have

$$E_{\delta,p_-,p_+}u=\big(p_+\big(1-e^{-2\delta}\big)\big(1-e^{-2}\big)+p_-\big(1+e^{-2\delta}\big)\big(1+e^{-2}\big)\big)u=e_{\delta,p_-,p_+}(1)u.$$

Example 8.1. Let g_- be identically equal to 1, $g_+ = 0$, $f_- = f_+ = \chi = 0$.

For $x \in]-1,0[$ we have

$$\begin{aligned}
u_-(x) = {}& \frac{1}{2e_{\delta,p_-,p_+}(1)}\Bigg[p_+\big(1-e^{-2\delta}\big)\Bigg[\big(e^{-x-2}-e^{x}\big)\int_{-1}^{0} e^{t}\,dt + \big(e^{x-1}-e^{-x-1}\big)\int_{-1}^{0} e^{-t-1}\,dt\Bigg] \\
&+ p_-\big(1+e^{-2\delta}\big)\Bigg[\big(e^{x}-e^{-x-2}\big)\int_{-1}^{0} e^{t}\,dt - \big(e^{x-1}+e^{-x-1}\big)\int_{-1}^{0} e^{-t-1}\,dt\Bigg]\Bigg] \\
&+ \frac{1}{2}\int_{-1}^{0} e^{-|x-t|}\,dt \\
= {}& \frac{1}{2e_{\delta,p_-,p_+}(1)}\big[p_+\big(1-e^{-2\delta}\big)\big(e^{-x-2}-e^{x}+e^{x-1}-e^{-x-1}\big) \\
&+ p_-\big(1+e^{-2\delta}\big)\big(e^{x}-e^{-x-2}-e^{-x-1}-e^{x-1}\big)\big]\big(1-e^{-1}\big) + \frac{2-e^{x}-e^{-x-1}}{2} \\
= {}& \frac{1}{2e_{\delta,p_-,p_+}(1)}\big[p_+\big(1-e^{-2\delta}\big)\big(1-e^{-1}\big)\big[e^{-x-2}-e^{x}+e^{x-1}-e^{-x-1} \\
&+ \big(2-e^{x}-e^{-x-1}\big)\big(1+e^{-1}\big)\big] \\
&+ p_-\big(1+e^{-2\delta}\big)\big[\big(e^{x}-e^{-x-2}-e^{-x-1}-e^{x-1}\big)\big(1-e^{-1}\big) \\
&+ \big(2-e^{x}-e^{-x-1}\big)\big(1+e^{-2}\big)\big]\big] \\
= {}& \frac{1}{e_{\delta,p_-,p_+}(1)}\big[p_+\big(1-e^{-2\delta}\big)\big(1-e^{-1}\big)\big(1-e^{x}-e^{-x-1}+e^{-1}\big) \\
&+ p_-\big(1+e^{-2\delta}\big)\big(1-e^{-x-1}-e^{x-1}+e^{-2}\big)\big] \\
= {}& \frac{1}{e_{\delta,p_-,p_+}(1)}\big[p_+\big(1-e^{-2\delta}\big)\big(1-e^{-1}\big)\big(1-e^{x}\big)\big(1-e^{-x-1}\big) \\
&+ p_-\big(1+e^{-2\delta}\big)\big(1-e^{-x-1}\big)\big(1-e^{x-1}\big)\big],
\end{aligned}$$

hence

$$\begin{aligned}
\frac{\|u_-\|_{L^p(]-1,0[;\mathbb{C})}}{\|g_-\|_{L^p(]-1,0[;\mathbb{C})}} \geqslant C\frac{1}{e_{\delta,p_-,p_+}(1)}&\Bigg[p_+\big(1-e^{-2\delta}\big)\Bigg(\int_{-1}^{0}\big(1-e^{x}\big)^p\big(1-e^{-x-1}\big)^p\,dx\Bigg)^{1/p} \\
&+ p_-\big(1+e^{-2\delta}\big)\Bigg(\int_{-1}^{0}\big(1-e^{-x-1}\big)^p\big(1-e^{x-1}\big)^p\,dx\Bigg)^{1/p}\Bigg]
\end{aligned}$$

$$\begin{aligned}&\geqslant C\frac{p_+(1-e^{-2\delta})+p_-(1+e^{-2\delta})}{p_+(1-e^{-2\delta})(1-e^{-2})+p_-(1+e^{-2\delta})(1+e^{-2})}\\&\geqslant C.\end{aligned}$$

Example 8.2. Let $g_-=0,\ g_+=0,\ f_-=1,\ f_+=\chi=0$.

For $x\in\,]-1,0[$ we have

$$u_-(x)=\frac{1}{e_{\delta,p_-,p_+}(1)}\big[p_+\big(1-e^{-2\delta}\big)\big(e^{-x-1}-e^{-1+x}\big)+p_-\big(1+e^{-2\delta}\big)\big(e^{-x-1}+e^{-1+x}\big)\big],$$

hence

$$\frac{\|u_-\|_{L^p(]-1,0[;\mathbb{C})}}{\|f_-\|}\geqslant C\frac{p_+(1-e^{-2\delta})+p_-(1+e^{-2\delta})}{e_{\delta,p_-,p_+}(1)}\geqslant C.$$

Example 8.3. Let $g_-=0,\ g_+=0,\ f_+=1,\ f_-=\chi=0$.

For $x\in\,]-1,0[$ we have

$$u_-(x)=\frac{1}{e_{\delta,p_-,p_+}(1)}2p_+e^{-\delta}\big(e^x-e^{-x-2}\big),$$

hence

$$\frac{\|u_-\|_{L^p(]-1,0[;\mathbb{C})}}{\|f_+\|}=\frac{2p_+e^{-\delta}}{e_{\delta,p_-,p_+}(1)}\left(\int_{-1}^{0}\big(e^x-e^{-x-2}\big)^p\,dx\right)^{1/p}\geqslant C\frac{p_+}{\delta p_++p_-}.$$

Example 8.4. Let $g_-=0,\ g_+=0,\ \chi=1,\ f_+=f_-=0$. For $x\in\,]-1,0[$ we have

$$u_-(x)=\frac{1}{e_{\delta,p_-,p_+}(1)}\big(1+e^{-\delta}\big)\big(e^x-e^{-x-2}\big),$$

hence

$$\frac{\|u_-\|_{L^p(]-1,0[;\mathbb{C})}}{\|\chi\|}=\frac{1+e^{-\delta}}{e_{\delta,p_-,p_+}(1)}\left(\int_{-1}^{0}\big(e^x-e^{-x-2}\big)^p\,dx\right)^{1/p}\geqslant C\frac{1}{\delta p_++p_-}.$$

Example 8.5. Let $g_-=0,\ g_+=0,\ f_-=1,\ f_+=\chi=0$.

For $x\in\,]0,\delta[$ we have

$$u_+(x)=\frac{2p_-}{e_{\delta,p_-,p_+}(1)}e^{-1}\big(e^{-x}+e^{x-2\delta}\big),$$

hence

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathbb{C})}}{\|f_-\|} \geqslant \frac{2p_-e^{-1}}{e_{\delta,p_-,p_+}(1)}\left(\int_0^\delta e^{-p\delta}\,dx\right)^{1/p} \geqslant C\frac{\delta^{1/p}p_-}{\delta p_+ + p_-}.$$

In Examples 8.6–8.10 we have $X=\ell^p$ and A is the operator such that

$$\mathcal{D}(A)=\left\{(u_n)_{n\in\mathbb{N}}\colon \left(n^2u_n\right)_{n\in\mathbb{N}}\in\ell^p\right\},$$
$$(Az)_n=-n^2u_n;$$

where $\mathbb{N}$ is the set of positive integers.

The space X has UMD property (see [27], Proposition 3) and it is easy to prove that A satisfies hypotheses H1, H2, H3, that the operator $B=(-A)^{1/2}$ is such that

$$\mathcal{D}(B)=\left\{(u_n)_{n\in\mathbb{N}}\colon (nu_n)_{n\in\mathbb{N}}\in\ell^p\right\},$$
$$(Bu)_n=nu_n$$

and that the semigroup V generated by $-B$ is such that

$$\left(V(x)u\right)_n=e^{-nx}u_n.$$

Moreover from [28], Theorem 1.18.5, it follows that

$$\left(\ell^p,\mathcal{D}(A)\right)_{\alpha,p}=\left\{(u_n)_{n\in\mathbb{N}}\in\ell^p : \left(n^{2\alpha}u_n\right)_{n\in\mathbb{N}}\in\ell^p\right\}$$

with the natural norm.

Example 8.6. Let $g_-=0$, $f_-=f_+=\chi=0$ and $g_+\in L^p(]0,\delta[;\ell^p)$ such that for a fixed $k\in\mathbb{N}$,

$$\left(g_+(x)\right)_n=\begin{cases} e^{-2kx/p} & \text{if } n=k,\\ 0 & \text{if } n\neq k.\end{cases}$$

We have

$$\|g_+\|_{L^p(]0,\delta[;\ell^p)}=\left(\int_0^\delta e^{-2kx}\,dx\right)^{1/p}=\left(\frac{1-e^{-2k\delta}}{2k}\right)^{1/p}.$$

For $x\in\,]-1,0[$ we have $(u_-(x))_n=0$ if $n\neq k$ and

$$\left(u_-(x)\right)_k=\frac{1}{ke_{\delta,p_-,p_+}(k)}p_+\left(e^{kx}-e^{-k(x+2)}\right)\int_0^\delta\left(e^{-kt}+e^{-k(2\delta-t)}\right)e^{-kt}\,dt$$
$$\geqslant\frac{1}{ke_{\delta,p_-,p_+}(k)}p_+\left(e^{kx}-e^{-k(x+2)}\right)\int_0^\delta e^{-2kt}\,dt$$

$$= \frac{1}{ke_{\delta,p_-,p_+}(k)} p_+\big(e^{kx} - e^{-k(x+2)}\big)\frac{1-e^{-2k\delta}}{2k},$$

hence

$$\begin{aligned}
\|Au_-\|_{L^p(]-1,0[;\ell^p)} &\geqslant \frac{p_+(1-e^{-2k\delta})}{2e_{\delta,p_-,p_+}(k)}\left(\int_{-1}^{0}\big(e^{kx} - e^{-k(x+2)}\big)^p\,dx\right)^{1/p} \\
&= \frac{p_+(1-e^{-2k\delta})}{k^{1/p}2e_{\delta,p_-,p_+}(k)}\left(\int_{-k}^{0}\big(e^{y} - e^{-y-2k}\big)^p\,dy\right)^{1/p} \\
&\geqslant \frac{p_+(1-e^{-2k\delta})}{k^{1/p}\,2e_{\delta,p_-,p_+}(k)}\left(\int_{-1}^{0}\big(e^{y} - e^{-y-2}\big)^p\,dy\right)^{1/p}.
\end{aligned}$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\frac{\|u_-\|_{L^p(]-1,0[;\mathcal{D}(A))}}{\|g_+\|_{L^p(]0,\delta[;\ell^p)}} \geqslant C\frac{p_+(1-e^{-2k\delta})^{1/p'}}{p_+(1-e^{-2k\delta})(1-e^{-2k}) + p_-(1+e^{-2k\delta})(1+e^{-2k})}.$$

If $k = [p_-/(\delta p_+)] + 2$ we have

$$\frac{p_-}{p_+} + \delta < k\delta \leqslant \frac{p_-}{p_+} + 2\delta,$$

hence

$$\begin{aligned}
\frac{\|u_-\|_{L^p(]-1,0[;\mathcal{D}(A))}}{\|g_+\|_{L^p(]0,\delta[;\ell^p)}} &\geqslant C\frac{p_+(1-e^{-2p_-/p_+-2\delta})^{1/p'}}{p_+(1-e^{-2p_-/p_+-4\delta}) + 4p_-} \\
&\geqslant C\frac{p_+}{p_+(p_-/p_+ + \delta) + p_-}\left(\frac{p_-/p_+ + \delta}{1 + p_-/p_+ + \delta}\right)^{1/p'} \\
&\geqslant C\frac{p_+}{\delta p_+ + p_-}\left(\frac{\delta p_+ + p_-}{p_+ + p_-}\right)^{1/p'} \\
&\geqslant C\frac{p_+}{(\delta p_+ + p_-)^{1/p}(p_+ + p_-)^{1/p'}}.
\end{aligned}$$

Example 8.7. Let $g_+ = 0$, $f_- = f_+ = \chi = 0$ and $g_- \in L^p(]-1,0[;\ell^p)$ such that for a fixed $k \in \mathbb{N}$,

$$\big(g_-(x)\big)_n = \begin{cases} e^{kx} & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We have

$$\|g_-\|_{L^p(]-1,0[;\ell^p)} = \left(\int_{-1}^{0} e^{pkx}\,dx\right)^{1/p} = \left(\frac{1-e^{-pk}}{pk}\right)^{1/p}.$$

For $x \in]0,\delta[$ we have $(u_+(x))_n = 0$ if $n \neq k$ and

$$\begin{aligned}
\big(u_+(x)\big)_k &= \frac{p_-}{ke_{\delta,p_-,p_+}(k)}\big(e^{-kx}+e^{-k(2\delta-x)}\big)\int_{-1}^{0}\big(e^{kt}-e^{-k(t+2)}\big)e^{kt}\,dt \\
&= \frac{p_-}{ke_{\delta,p_-,p_+}(k)}\big(e^{-kx}+e^{-k(2\delta-x)}\big)\frac{1-e^{-2k}-2ke^{-2k}}{2k},
\end{aligned}$$

hence

$$\begin{aligned}
\|Au_+\|_{L^p(]0,\delta[;\ell^p)} &\geqslant \frac{p_-(1-e^{-2k}-2ke^{-2k})}{2e_{\delta,p_-,p_+}(k)}\left(\int_0^{\delta} e^{-pkx}\right)^{1/p} \\
&= \frac{p_-(1-e^{-2k}-2ke^{-2k})}{2e_{\delta,p_-,p_+}(k)}\left(\frac{1-e^{-pk\delta}}{pk}\right)^{1/p}.
\end{aligned}$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|g_-\|_{L^p(]-1,0[;\ell^p)}} \geqslant C\frac{p_-(1-e^{-pk\delta})^{1/p}}{p_+(1-e^{-2k\delta})(1-e^{-2k})+p_-(1+e^{-2k\delta})(1+e^{-2k})}.$$

If $k = [p_-/(\delta p_+)] + 2$ we have

$$\frac{p_-}{p_+}+\delta < k\delta \leqslant \frac{p_-}{p_+}+2\delta,$$

hence

$$\begin{aligned}
\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|g_-\|_{L^p(]-1,0[;\ell^p)}} &\geqslant C\frac{p_-(1-e^{-pp_-/p_+-p\delta})^{1/p}}{p_+(1-e^{-2p_-/p_+-4\delta})+4p_-} \\
&\geqslant C\frac{p_-}{p_+(p_-/p_++\delta)+p_-}\left(\frac{p_-/p_++\delta}{1+p_-/p_++\delta}\right)^{1/p} \\
&\geqslant C\frac{p_-}{\delta p_++p_-}\left(\frac{\delta p_++p_-}{p_++p_-}\right)^{1/p} \\
&\geqslant C\frac{p_-}{(\delta p_++p_-)^{1/p'}(p_++p_-)^{1/p}}.
\end{aligned}$$

Example 8.8. Let $g_- = 0,\ f_- = f_+ = \chi = 0$ and $g_+ \in L^p(]0,\delta[;\ell^p)$ such that for a fixed $k \in \mathbb{N}$,

$$\big(g_+(x)\big)_n = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We have

$$\|g_+\|_{L^p(]0,\delta[;\ell^p)} = \left(\int_0^\delta 1\,dx\right)^{1/p} = \delta^{1/p}.$$

For $x \in]0,\delta[$ we have $(u_+(x))_n = 0$ if $n \neq k$ and

$$\begin{aligned}
\big(u_+(x)\big)_k &= \frac{1}{2ke_{\delta,p_-,p_+}(k)}\Bigg[p_+\big(1-e^{-2k}\big)\Bigg[\big(e^{-k(\delta-x)}+e^{-k(x+\delta)}\big)\int_0^\delta e^{-k(\delta-t)}\,dt \\
&\quad + \big(e^{-kx}+e^{-k(2\delta-x)}\big)\int_0^\delta e^{-kt}\,dt\Bigg] \\
&\quad + p_-\big(1+e^{-2k}\big)\Bigg[\big(e^{-k(\delta-x)}-e^{-k(x+\delta)}\big)\int_0^\delta e^{-k(\delta-t)}\,dt \\
&\quad - \big(e^{-kx}+e^{-k(2\delta-x)}\big)\int_0^\delta e^{-kt}\,dt\Bigg]\Bigg] + \frac{1}{2k}\int_0^\delta e^{-k|x-t|}\,dt \\
&= \frac{1}{2k^2e_{\delta,p_-,p_+}(k)}\big[p_+\big(1-e^{-2k}\big)\big(e^{-k(\delta-x)}+e^{-k(x+\delta)}+e^{-kx}+e^{-k(2\delta-x)}\big) \\
&\quad + p_-\big(1+e^{-2k}\big)\big(e^{-k(\delta-x)}-e^{-k(x+\delta)}-e^{-kx}-e^{-k(2\delta-x)}\big)\big]\big(1-e^{-k\delta}\big) \\
&\quad + \frac{2-e^{-kx}-e^{-k(\delta-x)}}{2k^2} \\
&= \frac{1}{2k^2e_{\delta,p_-,p_+}(k)}\big[p_+\big(1-e^{-2k}\big)\big(1-e^{-k\delta}\big)\big[e^{-kx}+e^{-k(2\delta-x)}+e^{-k(x+\delta)}+e^{-k(\delta-x)} \\
&\quad + \big(2-e^{-kx}-e^{-k(\delta-x)}\big)\big(1+e^{-k\delta}\big)\big] \\
&\quad + p_-\big(1+e^{-2k}\big)\big[\big(-e^{-kx}-e^{-k(2\delta-x)}-e^{-k(x+\delta)}+e^{-k(\delta-x)}\big)\big(1-e^{-k\delta}\big) \\
&\quad + \big(2-e^{-kx}-e^{-k(\delta-x)}\big)\big(1+e^{-2k\delta}\big)\big]\big] \\
&= \frac{1}{k^2e_{\delta,p_-,p_+}(k)}\big[p_+\big(1-e^{-2k}\big)\big(1-e^{-k\delta}\big)\big(1+e^{-k\delta}\big) \\
&\quad + p_-\big(1+e^{-2k}\big)\big(1-e^{-kx}-e^{-k(2\delta-x)}+e^{-2k\delta}\big)\big]
\end{aligned}$$

$$= \frac{1}{k^2 e_{\delta,p_-,p_+}(k)} \big[p_+\big(1-e^{-2k}\big)\big(1-e^{-2k\delta}\big) + p_-\big(1+e^{-2k}\big)\big(1-e^{-kx}\big)\big(1-e^{-k(2\delta-x)}\big)\big],$$

hence

$$\|Au_+\|_{L^p(]0,\delta[;\ell^p)} \geqslant \frac{1}{2^{1/p'} e_{\delta,p_-,p_+}(k)} \left[p_+\big(1-e^{-2k}\big)\big(1-e^{-2k\delta}\big)\left(\int_0^\delta 1\,dx\right)^{1/p} + p_-\big(1+e^{-2k}\big)\left(\int_0^\delta \big(1-e^{-kx}\big)^p\big(1-e^{-k(2\delta-x)}\big)^p\,dx\right)^{1/p}\right]$$

$$= \frac{1}{2^{1/p'} e_{\delta,p_-,p_+}(k)} \left[p_+\big(1-e^{-2k}\big)\big(1-e^{-2k\delta}\big)\delta^{1/p} + p_-\big(1+e^{-2k}\big)k^{-1/p}\left(\int_0^{k\delta} \big(1-e^{-x}\big)^p\big(1-e^{-(2k\delta-x)}\big)^p\,dx\right)^{1/p}\right].$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|g_+\|_{L^p(]0,\delta[;\ell^p)}} \geqslant C \frac{1}{p_+(1-e^{-2k\delta})(1-e^{-2k}) + p_-(1+e^{-2k\delta})(1+e^{-2k})} \times \left[p_+\big(1-e^{-2k\delta}\big) + p_- \frac{1}{(k\delta)^{1/p}}\left(\int_0^{k\delta} \big(1-e^{-x}\big)^p\big(1-e^{-(2k\delta-x)}\big)^p\,dx\right)^{1/p}\right].$$

If $k = [1/\delta] + 1$ we have

$$1 < k\delta \leqslant 1 + \delta,$$

hence

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|g_+\|_{L^p(]0,\delta[;\ell^p)}} \geqslant C \frac{1}{p_+ + 4p_-} \left[p_+\big(1-e^{-2}\big) + \frac{p_-}{(1+\delta)^{1/p}}\left(\int_0^1 \big(1-e^{-x}\big)^p\big(1-e^{-(2-x)}\big)^p\,dx\right)^{1/p}\right] \geqslant C.$$

Example 8.9. Let $g_- = 0$, $g_+ = 0$, $f_- = \chi = 0$ and $f_+ \in \ell^p$ such that for a fixed $k \in \mathbb{N}$,

$$f_{+n} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We have

$$\|f_+\|_{(\ell^p, \mathcal{D}(A))_{1/2-1/(2p),p}} = k^{1-1/p}.$$

For $x \in]0, \delta[$ we have $(u_+(x))_n = 0$ if $n \neq k$ and

$$\begin{aligned}\big(u_+(x)\big)_k = \frac{1}{k e_{\delta,p_-,p_+}(k)} \big[&p_+\big(1 - e^{-2k}\big)\big(e^{-k(\delta-x)} + e^{-k(x+\delta)}\big) \\ &+ p_-\big(1 + e^{-2k}\big)\big(e^{-k(\delta-x)} - e^{-k(x+\delta)}\big)\big]\end{aligned}$$

hence

$$\begin{aligned}\|Au_+\|_{L^p(]0,\delta[;\ell^p)} &\geqslant \frac{k}{2^{1/p'} e_{\delta,p_-,p_+}(k)} \Bigg[p_+\big(1 - e^{-2k}\big)\Bigg(\int_0^\delta \big(e^{-k(\delta-x)} + e^{-k(x+\delta)}\big)^p\,dx\Bigg)^{1/p} \\ &\quad + p_-\big(1 + e^{-2k}\big)\Bigg(\int_0^\delta \big(e^{-k(\delta-x)} - e^{-k(x+\delta)}\big)^p\,dx\Bigg)^{1/p}\Bigg] \\ &\geqslant \frac{k}{2^{1/p'} e_{\delta,p_-,p_+}(k)} \Bigg[p_+\big(1 - e^{-2k}\big)\Bigg(\int_0^\delta e^{-kp(\delta-x)}\,dx\Bigg)^{1/p} \\ &\quad + p_-\big(1 + e^{-2k}\big)\frac{e^{-k\delta}}{k^{1/p}}\Bigg(\int_0^{k\delta} \big(e^x - e^{-x}\big)^p\,dx\Bigg)^{1/p}\Bigg].\end{aligned}$$

We have

$$\begin{aligned}\int_0^y \sinh^p x\,dx &= \int_0^y \frac{\sinh^p x \cosh x}{\cosh x}\,dx \\ &= \frac{1}{p+1}\frac{\sinh^{p+1} y}{\cosh y} + \frac{1}{p+1}\int_0^y \frac{\sinh^{p+2} y}{\cosh^2 x}\,dx \\ &\geqslant \frac{1}{p+1}\frac{\sinh^{p+1} y}{\cosh y},\end{aligned}$$

hence there exists $C \in \mathbb{R}^+$ such that

$$\|Au_+\|_{L^p(]0,\delta[;\ell^p)} \geqslant C\frac{k}{e_{\delta,p_-,p_+}(k)}\left[p_+\left(1-e^{-2k}\right)\left(\frac{1-e^{-pk\delta}}{pk}\right)^{1/p} + p_-\frac{1+e^{-2k}}{k^{1/p}}\frac{(1-e^{-2k\delta})^{1+1/p}}{(1+e^{-2k\delta})^{1/p}}\right].$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|f_+\|_{(\ell^p,\mathcal{D}(A))_{1/2-1/(2p),p}}} \geqslant C\frac{p_+(1-e^{-pk\delta})^{1/p} + p_-(1-e^{-2k\delta})^{1+1/p}}{p_+(1-e^{-2k\delta})(1-e^{-2k}) + p_-(1+e^{-2k\delta})(1+e^{-2k})}.$$

If $k = [p_-/(\delta p_+)] + 2$ we have

$$\frac{p_-}{p_+} + \delta < k\delta \leqslant \frac{p_-}{p_+} + 2\delta,$$

hence

$$\begin{aligned}
&\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|f_+\|_{(\ell^p,\mathcal{D}(A))_{1/2-1/(2p),p}}} \\
&\quad\geqslant C\frac{p_+(1-\exp(-pp_-/p_+ - p\delta))^{1/p} + p_-(1-\exp(-2p_-/p_+ - 2\delta))^{1+1/p}}{p_+(1-\exp(-2p_-/p_+ - 4\delta)) + 4p_-} \\
&\quad\geqslant C\frac{1}{p_+(p_-/p_+ + \delta) + p_-}\left(p_+\left(\frac{p_-/p_+ + \delta}{1+p_-/p_+ + \delta}\right)^{1/p} + p_-\left(\frac{p_-/p_+ + \delta}{1+p_-/p_+ + \delta}\right)^{1+1/p}\right) \\
&\quad\geqslant C\frac{1}{\delta p_+ + p_-}\left(p_+\left(\frac{\delta p_+ + p_-}{p_+ + p_-}\right)^{1/p} + p_-\left(\frac{\delta p_+ + p_-}{p_+ + p_-}\right)^{1+1/p}\right) \\
&\quad\geqslant C\frac{p_+(p_+ + p_-) + p_-(\delta p_+ + p_-)}{(\delta p_+ + p_-)^{1/p'}(p_+ + p_-)^{1+1/p}} \\
&\quad\geqslant C\left(\frac{p_+ + p_-}{\delta p_+ + p_-}\right)^{1/p'}.
\end{aligned}$$

Example 8.10. Let $g_- = 0$, $g_+ = 0$, $f_- = f_+ = 0$ and $\chi \in \ell^p$ such that for a fixed $k \in \mathbb{N}$,

$$\chi_n = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We have

$$\|\chi\|_{(\ell^p,\mathcal{D}(A))_{1/2-1/(2p),p}} = k^{1-1/p}.$$

For $x \in]0,\delta[$ we have $(u_+(x))_n = 0$ if $n \neq k$ and

$$\left(u_+(x)\right)_k = \frac{1}{ke_{\delta,p_-,p_+}(k)}\left(e^{-kx} + e^{-k(2\delta - x)}\right)\left(1-e^{-2k}\right)$$

hence

$$\|Au_+\|_{L^p(]0,\delta[;\ell^p)} \geqslant \frac{k}{e_{\delta,p_-,p_+}(k)}\big(1-e^{-2k}\big)\left(\int_0^\delta e^{-pkx}\right)^{1/p}$$
$$= \frac{k}{e_{\delta,p_-,p_+}(k)}\big(1-e^{-2k}\big)\left(\frac{1-e^{-pk\delta}}{pk}\right)^{1/p}.$$

Therefore there exists $C \in \mathbb{R}^+$ such that

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|\chi\|_{(\ell^p,\mathcal{D}(A))_{1/2-1/(2p),p}}} \geqslant C\frac{(1-e^{-2k})(1-e^{-pk\delta})^{1/p}}{p_+(1-e^{-2k\delta})(1-e^{-2k})+p_-(1+e^{-2k\delta})(1+e^{-2k})}.$$

If $k = [p_-/(\delta p_+)] + 2$ we have

$$\frac{p_-}{p_+} + \delta < k\delta \leqslant \frac{p_-}{p_+} + 2\delta,$$

hence

$$\frac{\|u_+\|_{L^p(]0,\delta[;\mathcal{D}(A))}}{\|\chi\|_{(\ell^p,\mathcal{D}(A))_{1/2-1/(2p),p}}} \geqslant C\frac{(1-\exp(-pp_-/p_+-p\delta))^{1/p}}{p_+(1-\exp(-2p_-/p_+-4\delta))+4p_-}$$
$$\geqslant C\frac{1}{p_+(p_-/p_++\delta)+p_-}\left(\frac{p_-/p_++\delta}{1+p_-/p_++\delta}\right)^{1/p}$$
$$\geqslant C\frac{1}{(\delta p_++p_-)^{1/p'}(p_++p_-)^{1/p}}.$$

9. Applications

We apply Theorems 6.1 and 7.1 to the problem

$$\begin{cases} \Delta u_-(x,y) = -g_-(x,y), & (x,y) \in\,]-1,0[\, \times \Omega, \\ \Delta u_+(x,y) = -g_+(x,y), & (x,y) \in\,]0,\delta[\, \times \Omega, \\ u_-(-1,y) = f_-(y), & y \in \Omega, \\ \dfrac{\partial u_+}{\partial x}(\delta,y) = f_+(y), & y \in \Omega, \\ u_-(x,y) = 0, & x \in\,]-1,0[,\ y \in \partial\Omega, \\ u_+(x,y) = 0, & x \in\,]0,\delta[,\ y \in \partial\Omega, \\ u_-(0,y) = u_+(0,y), & y \in \Omega, \\ p_-\dfrac{\partial u_-}{\partial x}(0,y) = p_+\dfrac{\partial u_+}{\partial x}(0,y), & y \in \Omega, \end{cases} \tag{9.1}$$

introduced in Example 2.3.

We suppose that Ω is an open bounded subset of $\mathbb{R}^2$, with C^∞ boundary. Problem (9.1) can be written in the abstract form (ATP) (with $\chi = 0$) if we put $X = L^p(\Omega)$ and A is the realisation of the Laplace operator in $L^p(\Omega)$ with Dirichlet boundary conditions, that is

$$A : \mathcal{D}(A) = \left\{u \in W^{2,p}(\Omega)\colon\ u|_{\partial\Omega} = 0\right\} \to L^p(\Omega),$$
$$Au = \Delta u.$$

The space X has UMD property (see [27], Proposition 3); the operator $-A$ invertible with bounded inverse (see [18], Theorem 9.15 and Lemma 9.17), is sectorial and has bounded H^∞ functional calculus (see [13], Theorem 4 or [22], Section 7). Hence the hypotheses of Theorems 6.1 and 7.1 are fulfilled.

If I is an open interval of R, then we have $L^p(I, W^{2,p}(\Omega)) \cap W^{2,p}(I, L^p(\Omega)) = W^{2,p}(I \times \Omega)$ (see [28], Theorem 4.2.4); therefore from Theorems 6.1 and 7.1 it follows that for every $g_- \in L^p(]-1,0[\times \Omega)$, $g_+ \in L^p(]0,\delta[\times \Omega)$, $f_- \in (X, \mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X, \mathcal{D}(A))_{1/2-1/(2p),p}$ problem (9.1) has one and only one solution

$$(u_-, u_+) \in W^{2,p}\big(]-1,0[\times \Omega\big) \times W^{2,p}\big(]0,\delta[\times \Omega\big).$$

Note that we have

$$\big(X, \mathcal{D}(A)\big)_{1-1/(2p),p} = \left\{u \in W^{2-1/p,p}(\Omega)\colon\ u|_{\partial\Omega} = 0\right\},$$
$$\big(X, \mathcal{D}(A)\big)_{1/2-1/(2p),p} = \begin{cases} \{u \in W^{1-1/p,p}(\Omega)\colon\ u|_{\partial\Omega} = 0\} & \text{if } p > 2, \\ W^{1-1/p,p}(\Omega) & \text{if } 1 < p < 2 \end{cases}$$

(see [28], Theorem 4.3.3).

Now we study the problem

$$\begin{cases} \Delta u_-(x,y) = -g_-(x,y), & (x,y) \in]-1,0[\times]-\pi,\pi[, \\ \Delta u_+(x,y) = -g_+(x,y), & (x,y) \in]0,\delta[\times]-\pi,\pi[, \\ u_-(-1,y) = f_-(y), & y \in]-\pi,\pi[, \\ \dfrac{\partial u_+}{\partial x}(\delta,y) = f_+(y), & y \in]-\pi,\pi[, \\ u_-(x,\pi) = u_-(x,-\pi), & x \in]-1,0[, \\ u_+(x,\pi) = u_+(x,-\pi), & x \in]0,\delta[, \\ \dfrac{\partial u_-}{\partial x}(x,\pi) = \dfrac{\partial u_-}{\partial x}(x,-\pi), & x \in]-1,0[, \\ \dfrac{\partial u_+}{\partial x}(x,\pi) = \dfrac{\partial u_+}{\partial x}(x,-\pi), & x \in]0,\delta[, \\ u_-(0,y) = u_+(0,y), & y \in]-\pi,\pi[, \\ p_-\dfrac{\partial u_-}{\partial x}(0,y) = p_+\dfrac{\partial u_+}{\partial x}(0,y), & y \in]-\pi,\pi[\end{cases} \tag{9.2}$$

introduced in Example 2.2.

Problem (9.2) can be written in the abstract form (ATP) (with $\chi = 0$) if $X = L^p_{\mathrm{per}}(0, 2\pi)$, the space of 2π-periodic measurable functions on $\mathbb{R}$ with locally summable p-th power, with norm

$$\|f\|_X = \left(\int_0^{2\pi} \bigl|f(x)\bigr|^p \, dx \right)^{1/p},$$

and A is the operator in X such that

$$\mathcal{D}(A) = W^{2,p}_{\mathrm{per}}(0, 2\pi) = \bigl\{u \in X\colon\ u'' \in X\bigr\},$$
$$Au = u'',$$

where the derivative is in the sense of distributions.

It is easy to check that A is closed, densely defined and if $\lambda \notin \{n^2\colon\ n \in \mathbb{N} \cup \{0\}\}$ then $\lambda \in \rho(-A)$ with

$$\bigl((\lambda I + A)^{-1} g\bigr)(x) = \int_{x-2\pi}^{x} \frac{\cos(\lambda^{1/2}(x - t - \pi))}{2\lambda^{1/2} \sin(\pi \lambda^{1/2})} g(t)\, dt = \int_0^{2\pi} \frac{\cos(\lambda^{1/2}(t - \pi))}{2\lambda^{1/2} \sin(\pi \lambda^{1/2})} g(x - t)\, dt;$$

we note that the integral kernel does not depend on the choice of the determination of $\lambda^{1/2}$. Therefore, by Minkowski integral inequality,

$$\begin{aligned}
\bigl\|(\lambda I + A)^{-1} g\bigr\|_X &\leqslant \left(\int_0^{2\pi} \left(\int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t - \pi))|}{2|\lambda|^{1/2} |\sin(\pi \lambda^{1/2})|} \bigl|g(x - t)\bigr|\, dt \right)^p dx \right)^{1/p} \\
&\leqslant \int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t - \pi))|}{2|\lambda|^{1/2} |\sin(\pi \lambda^{1/2})|} \left(\int_0^{2\pi} \bigl|g(x - t)\bigr|^p\, dx \right)^{1/p} dt \\
&= \int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t - \pi))|}{2|\lambda|^{1/2} |\sin(\pi \lambda^{1/2})|}\, dt \|g\|_X;
\end{aligned}$$

hence

$$\bigl\|(\lambda I + A)^{-1}\bigr\| \leqslant \int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t - \pi))|}{2|\lambda|^{1/2} |\sin(\pi \lambda^{1/2})|}\, dt.$$

We have

$$|\cos z| \leqslant \frac{|e^{iz}| + |e^{-iz}|}{2} = \cosh(\operatorname{Im} z), \qquad |\sin z| \geqslant \left| \frac{|e^{iz}| - |e^{-iz}|}{2} \right| = \bigl|\sinh(\operatorname{Im} z)\bigr|,$$

therefore, if $\lambda \notin \mathbb{R}^- \cup \{0\}$, we have

$$\int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t-\pi))|}{2|\lambda|^{1/2}|\sin(\pi\lambda^{1/2})|}\,dt \leqslant \int_0^{2\pi} \frac{\cosh(\operatorname{Im}\lambda^{1/2}(t-\pi))}{2|\lambda|^{1/2}|\sinh(\pi \operatorname{Im}\lambda^{1/2})|}\,dt = \frac{1}{|\operatorname{Im}\lambda^{1/2}||\lambda|^{1/2}}.$$

From this estimate it follows easily that $-A$ is a sectorial operator with spectral angle 0, hence H1 is satisfied.

The operator A is not invertible, since 0 is an isolated eigenvalue whose eigenfunctions are the constant functions. Therefore Theorems 6.1 and 7.1 cannot be applied directly.

We have

$$\operatorname{Res}\left(\frac{\cos(\lambda^{1/2}(t-\pi))}{2\lambda^{1/2}\sin(\pi\lambda^{1/2})}, \lambda = 0\right) = \frac{1}{2\pi},$$

hence the spectral projection corresponding to the eigenvalue 0 is the operator P such that

$$(Pu)(x) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt.$$

The space X can be split into the direct sum of the image and the kernel of P, that is the direct sum of the space $X_0 = \ker P$ of zero mean functions and the space X_1 of constant functions. Hence every solution (u_-, u_+) of (ATP) can be written in the form $(v_- + w_-, v_+ + w_+)$, with (v_-, v_+) solution of the following problem in the space X_0,

$$\begin{cases} v_-''(x) + A_0 v_-(x) = -(I-P)g_-(x), & x \in\,]-1, 0[,\\ v_+''(x) + A_0 v_+(x) = -(I-P)g_+(x), & x \in\,]0, \delta[,\\ v_-(-1) = (I-P)f_-, \\ v_+'(\delta) = (I-P)f_+, \\ v_-(0) = v_+(0), \\ p_- v_-'(0) = p_+ v_+'(0), \end{cases} \tag{9.3}$$

where A_0 is the part of A in X_0, and (w_-, w_+) solution of the following problem in the one dimensional space X_1,

$$\begin{cases} w_-''(x) = -Pg_-(x), & x \in\,]-1, 0[,\\ w_+''(x) = -Pg_+(x), & x \in\,]0, \delta[,\\ w_-(-1) = Pf_-, \\ w_+'(\delta) = Pf_+, \\ w_-(0) = w_+(0), \\ p_- w_-'(0) = p_+ w_+'(0). \end{cases} \tag{9.4}$$

From the standard properties of spectral projections it follows that $\mathcal{D}(A_0) = \mathcal{D}(A) \cap X_0$, A_0 is invertible, $\rho(A_0) = \rho(A) \cup \{0\}$ and for $\lambda \in \rho(A)$ we have $\|(\lambda I - A_0)^{-1}\| \leqslant \|(\lambda I - A)^{-1}\|$, therefore A_0 satisfies H1 and H2.

Now we prove that $-A_0$ has bounded H^∞ functional calculus. Let $\varphi \in\,]0, \pi[$, $f \in H_0^\infty(S_\varphi)$ and let $C, s \in \mathbb{R}^+$ be such that $|f(z)| \leqslant C|z|^{-s}$, for $z \in S_\varphi$. For $\alpha \in\,]0, \varphi[$ and $n \in \mathbb{N}$, let $\Gamma_{\alpha,n}$ be the path composed by the part of the parabola $\{z^2\colon \operatorname{Re} z = n + 1/2\}$ contained in the sector S_α and the part of Γ_α on the right of the parabola, that is

$$\Gamma_{\alpha,n} = \left\{\rho e^{\pm i\alpha}\colon \rho \geqslant 2\left(n+\frac{1}{2}\right)^2 \frac{1-\cos\alpha}{\sin^2\alpha}\right\} \cup \left\{\left(n+\frac{1}{2}+i\rho\right)^2\colon |\rho| \leqslant \left(n+\frac{1}{2}\right)\frac{1-\cos\alpha}{\sin\alpha}\right\}.$$

Then we have

$$\begin{aligned} f(-A_0) &= \frac{1}{2\pi i}\int_{\Gamma_\alpha} f(z)(zI+A_0)^{-1}\,dz \\ &= \sum_{j=1}^{n} \operatorname{Res}\bigl(f(z)(zI+A_0)^{-1},\ z=j^2\bigr) + \frac{1}{2\pi i}\int_{\Gamma_{\alpha,n}} f(z)(zI+A_0)^{-1}\,dz. \end{aligned}$$

The last addend converges to 0 as n goes to ∞. This is obvious for the integral on the two half-lines

$$\left\{\rho e^{\pm i\alpha}\colon \rho \geqslant 2\left(n+\frac{1}{2}\right)^2 \frac{1-\cos\alpha}{\sin^2\alpha}\right\}.$$

Moreover if $\lambda = ((n+1/2)+i\rho)^2$, then, taking into account Proposition 4.10,

$$\begin{aligned} \int_0^{2\pi} \frac{|\cos(\lambda^{1/2}(t-\pi))|}{2|\lambda|^{1/2}|\sin(\pi\lambda^{1/2})|}\,dt &\leqslant \int_0^{2\pi} \frac{\cosh(\rho(t-\pi))}{2\sqrt{(n+1/2)^2+\rho^2}\cosh(\pi\rho)}\,dt \\ &= \frac{\sinh(\pi\rho)}{\sqrt{(n+1/2)^2+\rho^2}\,\rho\cosh(\pi\rho)} \\ &\leqslant \frac{4\pi}{\sqrt{(n+1/2)^2+\rho^2}(1+2\pi|\rho|)}. \end{aligned}$$

Therefore the norm of the integral of $f(z)(zI+A_0)^{-1}$ on the arc of parabola contained in $\Gamma_{\alpha,n}$ is less or equal than

$$\begin{aligned} &\int_{-(n+\frac{1}{2})\frac{1-\cos\alpha}{\sin\alpha}}^{(n+\frac{1}{2})\frac{1-\cos\alpha}{\sin\alpha}} \frac{4}{\sqrt{(n+1/2)^2+\rho^2}(1+2\pi|\rho|)} \left|f\left(\left(n+\frac{1}{2}+i\rho\right)^2\right)\right| 2\left|n+\frac{1}{2}+i\rho\right|\,d\rho \\ &\quad\leqslant 8C\left(n+\frac{1}{2}\right)^{-s} \int_{-(n+\frac{1}{2})\frac{1-\cos\alpha}{\sin\alpha}}^{(n+\frac{1}{2})\frac{1-\cos\alpha}{\sin\alpha}} \frac{1}{1+2\pi|\rho|}\,d\rho \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Therefore

$$f(-A_0)=\sum_{j=1}^{\infty}\operatorname{Res}\bigl(f(z)(zI+A_0)^{-1},\ z=j^2\bigr).$$

Since $\lim_{\lambda\to j^2}(\lambda-j^2)/\sin(\pi\lambda^{1/2})=2j(-1)^j/\pi$, we have

$$\operatorname{Res}\biggl(\frac{\cos(\lambda^{1/2}(t-\pi))}{2\lambda^{1/2}\sin(\pi\lambda^{1/2})},\lambda=j^2\biggr)=\frac{\cos(jt)}{\pi};$$

hence

$$\begin{aligned}\bigl(f(-A_0)g\bigr)(x)&=\sum_{j=1}^{\infty}\frac{f(j^2)}{\pi}\int_0^{2\pi}\cos\bigl(j(x-t)\bigr)g(t)\,dt\\&=\sum_{j=1}^{\infty}\biggl(\frac{f(j^2)}{2\pi}\int_0^{2\pi}e^{-jt}g(t)\,dt\,e^{jx}+\frac{f(j^2)}{2\pi}\int_0^{2\pi}e^{jt}g(t)\,dt\,e^{-jx}\biggr).\end{aligned}$$

Therefore, by Marcinkiewicz multiplier theorem (see [14], Theorem 8.2.1), there exists $C\in\mathbb{R}^+$ such that

$$\bigl\|f(-A_0)\bigr\|\leqslant C\biggl(\sup_{j\in\mathbb{N}}\bigl|f\bigl(j^2\bigr)\bigr|+\sup_{j\in\mathbb{N}}\sum_{\ell=2^{j-1}}^{2^j-1}\bigl|f\bigl((\ell+1)^2\bigr)-f\bigl(\ell^2\bigr)\bigr|\biggr).$$

Obviously $\sup_{j\in\mathbb{N}}|f(j^2)|\leqslant\|f\|_\infty$. In order to estimate $\sup_{j\in\mathbb{N}}\sum_{\ell=2^{j-1}}^{2^j-1}|f((\ell+1)^2)-f(\ell^2)|$ we observe that, if $z\in\mathbb{R}^+$ and $0<\sigma<\sin\varphi$, then $C_{\sigma z}(z)$, the circle of centre z and radius σz, is included in S_φ, therefore

$$\bigl|f'(z)\bigr|=\biggl|\frac{1}{2\pi i}\int_{C_{\sigma z}(z)}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\biggr|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(z+\sigma ze^{i\theta})|}{(\sigma z)^2}\sigma z\,d\theta\leqslant\frac{\|f\|_\infty}{\sigma z}.$$

Hence

$$\sum_{\ell=2^{j-1}}^{2^j-1}\bigl|f\bigl((\ell+1)^2\bigr)-f\bigl(\ell^2\bigr)\bigr|\leqslant\int_{2^{2j-2}}^{2^{2j}}\bigl|f'(z)\bigr|\,dz\leqslant\int_{2^{2j-2}}^{2^{2j}}\frac{\|f\|_\infty}{\sigma z}\,dz=\frac{\log 4}{\sigma}\|f\|_\infty.$$

We have proved that there exists $C\in\mathbb{R}^+$ such that for every $f\in H_0^\infty(S_\varphi)$ we have $\|f(-A_0)\|\leqslant C\|f\|_\infty$, hence, by Proposition 3.1, $-A_0$ has bounded $H^\infty(S_\varphi)$ functional calculus and H3 is satisfied.

Therefore A_0 satisfies the hypotheses of Theorems 6.1 and 7.1. Since the operator $I-P$ is continuous from X to X_0 and from $\mathcal{D}(A)$ to $\mathcal{D}(A_0)$, it transforms elements of $(X,\mathcal{D}(A))_{\theta,p}$ into elements of $(X_0, \mathcal{D}(A_0))_{\theta,p}$; hence for every $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_- \in (X,\mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X,\mathcal{D}(A))_{1/2-1/(2p),p}$ problem (9.3) has one and only one solution.

It is easy to check that the only solution of problem (9.4) is $(w_-, w_+) \in W^{2,p}(]-1,0[, X_1) \times W^{2,p}(]0,\delta[, X_1)$ such that

$$w_-(x) = Pf_- + \frac{p_+}{p_-}(x+1)Pf_+ + \int_{-1}^{0} \frac{x+t+2-|x-t|}{2} Pg_-(t)\,dt + \frac{p_+}{p_-}(x+1)\int_0^{\delta} Pg_+(t)\,dt,$$

$$w_+(x) = Pf_- + \left(\frac{p_+}{p_-} + x\right)Pf_+ + \int_{-1}^{0}(t+1)Pg_-(t)\,dt + \int_0^{\delta}\left(\frac{p_+}{p_-} + \frac{x+t-|x-t|}{2}\right)Pg_+(t)\,dt.$$

Hence problem (9.2) has one and only one solution for every $g_- \in L^p(]-1,0[;X)$, $g_+ \in L^p(]0,\delta[;X)$, $f_- \in (X,\mathcal{D}(A))_{1-1/(2p),p}$, $f_+ \in (X,\mathcal{D}(A))_{1/2-1/(2p),p}$, but the dependence of the norm of the solution on δ, p_- and p_+ is in general worse than that obtained in Theorems 6.1 and 7.1 (recall that in this example $0 \notin \rho(A)$).

Indeed, from the representation of w_- and w_+ we obtain the following estimates

$$\|w_-\|_{L^p(]-1,0[)} \leqslant C\left(\|Pf_-\| + \frac{p_+}{p_-}\|Pf_+\| + \|Pg_-\|_{L^p(]-1,0[)} + \frac{\delta^{1/p'}p_+}{p_-}\|Pg_+\|_{L^p(]0,\delta[)}\right),$$

$$\|w_-''\|_{L^p(]-1,0[)} = \|Pg_-\|_{L^p(]-1,0[)},$$

$$\|w_+\|_{L^p(]0,\delta[)} \leqslant C\left(\delta^{1/p}\|Pf_-\| + \delta^{1/p}\left(\frac{p_+}{p_-}+\delta\right)\|Pf_+\| + \delta^{1/p}\|Pg_-\|_{L^p(]-1,0[)} + \delta\left(\frac{p_+}{p_-}+\delta\right)\|Pg_+\|_{L^p(]0,\delta[)}\right),$$

$$\|w_+''\|_{L^p(]0,\delta[)} = \|Pg_+\|_{L^p(]0,\delta[)};$$

where the dependence on δ, p_- and p_+ is sharp. These estimates are easily obtained, with the exception of the estimate of $\|w_+\|$ in terms of $\|Pf_+\|$ and $\|Pg_+\|$. We have

$$\left(\int_0^{\delta}\left\|\left(\frac{p_+}{p_-}+x\right)Pf_+\right\|^p dx\right)^{1/p} = \frac{1}{(p+1)^{1/p}}\left(\left(\frac{p_+}{p_-}+\delta\right)^{p+1} - \left(\frac{p_+}{p_-}\right)^{p+1}\right)^{1/p}\|Pf_+\|;$$

but

$$\left(\frac{p_+}{p_-}+\delta\right)^{p+1}-\left(\frac{p_+}{p_-}\right)^{p+1}\geqslant\left(\frac{p_+}{p_-}+\delta\right)^{p+1}-\frac{p_+}{p_-}\left(\frac{p_+}{p_-}+\delta\right)^{p}=\delta\left(\frac{p_+}{p_-}+\delta\right)^{p}$$

and, by applying the mean value theorem to the function $t\mapsto(p_+/p_-+t)^{p+1}$ on the interval $[0,\delta]$, we get

$$\left(\frac{p_+}{p_-}+\delta\right)^{p+1}-\left(\frac{p_+}{p_-}\right)^{p+1}\leqslant(p+1)\delta\left(\frac{p_+}{p_-}+\delta\right)^{p}.$$

Moreover we have

$$\begin{aligned}
&\left(\int_0^\delta\left\|\int_0^\delta\left(\frac{p_+}{p_-}+\frac{x+t-|x-t|}{2}\right)Pg_+(t)\,dt\right\|^p dx\right)^{1/p}\\
&\quad\leqslant\frac{p_+}{p_-}\left(\int_0^\delta\left\|\int_0^\delta Pg_+(t)\,dt\right\|^p dx\right)^{1/p}\\
&\qquad+\left(\int_0^\delta\left(\int_0^\delta\left|\frac{x+t-|x-t|}{2}\right|^{p'}dt\right)^{p/p'}\left(\int_0^\delta\|Pg_+(t)\|^p\,dt\right)dx\right)^{1/p}\\
&\quad\leqslant\left[\delta\frac{p_+}{p_-}+\delta^2\left(\int_0^1\left(\int_0^1\left|\frac{y+s-|y-s|}{2}\right|^{p'}ds\right)^{p/p'}dy\right)^{1/p}\right]\|Pg_+\|_{L^p(]0,\delta[)}.
\end{aligned}$$

The dependence on δ, p_- and p_+ in this estimate is sharp, as can be seen by considering the case in which Pg_+ is constant with respect to t.

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